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An Eightfold Littlewood-Richardson Theorem

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Abstract. Based on a generalized ordering $\propto$ on a set X, Schensted's insertion mapping is defined on the set of words $(W, \cdot)$ over the ordered alphabet $(X, \propto)$. In this general framework, a transparent approach to various versions of the Robinson-Schensted correspondence and of invariant properties originally due to Schützenberger, Knuth, White e.a. is obtained. Furthermore, eight combinatorial descriptions of the Littlewood-Richardson coefficients are obtained simultaneously, and direct bijections between the corresponding sets, including the bijection of Hanlon and Sundaram. Some of these descriptions may be translated into identities of skew Schur functions discovered by Aitken and Berenstein/Zelevinsky.

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Introduction

Let $\mathbb{N}$ ($\mathbb{N}_0$, resp.) be the set of all positive (nonnegative, resp.) integers and

$$\underline{z} := \{\, n \in \mathbb{N} \,|\, n \leq z \,\}$$

for all integers z. In a remarkable paper of 1961 [13], Schensted discovered an algorithmic way to determine the length of the longest increasing and decreasing subsequence of a finite sequence over $\mathbb{N}$, which, in the meantime, turned out to be of crucial importance for the representation theory of the symmetric groups. It is essentially based on the following *insertion mappings*: Let w be a nondecreasing word over the alphabet $\mathbb{N}$, that is, an element $w = w_1 \cdots w_n$ of a free monoid $(\mathcal{W}, \cdot)$ over $\mathbb{N}$ such that $w_1 \leq \cdots \leq w_n$. Then, for any $x \in \mathbb{N}$, let

$$w \boxplus x := \begin{cases} w & \text{if } w_n \leq x \\ w_1 \cdots w_{j-1} x w_{j+1} \cdots w_n & \text{if } x < w_n \end{cases} \in \mathcal{W}$$

and

$$w \underline{\angle} x := \begin{cases} x & \text{if } w_n \leq x \\ w_j & \text{if } x < w_n \end{cases} \in \mathbb{N}\,,$$

where, in the case of $x < w_n$, the index $j \in \underline{n}$ is defined by the condition $w_{j-1} \leq x < w_j$. In this interesting second case, the word w and the letter x may be recovered from $v = v_1 \cdots v_n := w \boxplus x$ and $y := w \nleqslant x$, as $v_{j+1} \geq y > v_j$. This is the core of Schensted's bijection.

Two observations served as the main stimulus for the present paper: Denoting by $\bar{\cdot}$ the inverse product on $\mathcal{W}$, we obtain a second pair of insertion mappings for the (free) monoid $(\mathcal{W}, \bar{\cdot})$ and the ordering $\geq$, which are denoted by $x \boxminus w$ instead of $w \boxplus x$ and $x \ngeqslant w$ instead of $w \nleqslant x$. Surprisingly, these mappings are the inverse mappings of the insertion mappings described above. In other words, Schensted's *deletion algorithm* is also an *insertion algorithm*, namely the one given by the inverse product and the inverse ordering. More formally, in the case of $x < w_n$, we have:

$$(w \nleqslant x) \boxminus (w \boxplus x) = w \quad \text{and} \quad (w \nleqslant x) \ngeqslant (w \boxplus x) = x.$$

This was the first of the two starting points of our investigations. Viewing tableaux as elements of a free monoid $(\mathcal{T}, \bullet)$ over $\mathcal{W}$, the insertion mappings $\boxplus$ and $\nleqslant$ may be extended naturally to the set of tableaux by induction. Again, considering the inverse product $\bar{\bullet}$ on $\mathcal{T}$, we obtain the corresponding inverse mappings. This leads to a short proof of the Robinson-Schensted correspondence

$$(P, Q) : W \longrightarrow \bigcup_{r \text{ Partition}} S\overset{\leq}{\mathcal{T}}{}^r \times \mathcal{L}_r, \tag{1}$$

Here, in the first component, we obtain as the P-symbol of w a standard tableau $t \in \mathcal{T}$, which is increasing in rows and strictly increasing in columns, while the Q-symbol in the second component is a standard word (or: lattice permutation). As is easily seen, for the bijection (1), instead of $\leq$, a more general relation $\propto$ may be used (Section 1). For example, in the special case of $\propto = <$, Knuth's *dual correspondence* is obtained [7]. This was the second starting point of this paper.

The bijection (1), now for the relation $\propto$ instead of $\leq$, may be refined upon significantly by considering certain invariants. For any $w = w_1 \cdots w_n \in \mathcal{W}$, let

$$\mathcal{D}_\propto(w) = \{\, i \in \underline{n-1} \mid w_i \not\propto w_{i+1} \,\} ,$$

where $x \not\propto y$ means that $x \propto y$ does not hold. $\mathcal{D}_\propto(w)$ is called the $\propto$-*descent set* of w. In Section 2, it is particularly shown that $\mathcal{D}_\propto(w) = \mathcal{D}_\geq(Q(w))$, for all $w \in \mathcal{W}$ (Theorem 2). In the special case of $\propto = \leq$, this result is due to Schützenberger ([14], Remarque 2, see also [3], Theorem 2.1). For the dual correspondence ($\propto = <$) we obtain a result due to Knuth ([7], Theorem 1*). The descent set may be generalized by the property of a word w to be a shifted $\propto$-filling, which is also transferred by the Q-symbol (Theorem (3)). This fairly intricate result enables us to give several descriptions of the Littlewood-Richardson coefficients as will be described below. As a special case, we obtain here Theorem 1 of [17].

In Section 4, our combinatorial investigations are completed by introducing and analyzing the notion of a conjugate and rotated tableau. It leads to our main combinatorial result (Main Theorem 1).

The Littlewood-Richardson (L-R) coefficient c^u_{qp} describes the multiplicity of the irreducible character ζ^u of the symmetric group S_{n+k} in the outer product of the irreducible characters ζ^q of S_n and ζ^p of S_k. In order to describe them combinatorially, we use the following characterization (Corollary 4):

For all partitions p of k, q of n and u of $n+k$, let C^u_{qp} be a set such that there exists a bijection

$$S^q \overset{\leq}{\mathcal{T}}{}^{u-q}_p \longrightarrow \bigcup_r S\overset{\leq}{\mathcal{T}}{}^r_p \times C^u_{qr}, \tag{2}$$

where the union is taken over all partitions r of k. Then we have $c^u_{qp} = |C^u_{qp}|$ for all q, p, u.

Taking into account the above mentioned invariants of the Q-symbol, bijections of this kind are established by the Robinson-Schensted correspondence (1). It may therefore be viewed as a combinatorial core of the representation theory of the symmetric groups. This concept for proving the L-R rule has been used in in [11], [16] and [17]. In each of these approaches, the case where $\propto = <$ is considered.

More generally, in Section 5, considering $\propto \in \{\leq, <, \geq, >\}$ (and a variation of the Q-symbol), eight combinatorial descriptions for the L-R coefficients are obtained simultaneously: c^u_{qp} is equal to the number of lattice permutations of content p (p^*, resp.), which, row- or column-wise, fit into the (conjugate, rotated) skew diagram corresponding to q and u.

A short additional analysis of the bijections (2) in Section 6 shows that the Robinson-Schensted correspondence yields direct bijections between the eight L-R sets by fixing the P-symbol. As a special case, this includes the bijection introduced in [5].

1 Schensted's insertion mappings

In the sequel, an arbitrary set X instead of $\mathbb{N}$ as in our introduction will be considered. To start with, let us fix some notation. A relation $\propto$ on X is called an *almost complete ordering* (on X), if

(a) $\propto$ is anti-symmetric ($x \propto y$, $y \propto x \Longrightarrow x = y$)

(b) $\propto$ is transitive ($x \propto y$, $y \propto z \Longrightarrow x \propto z$)

(c) $\propto$ is complete ($x \neq y \Longrightarrow x \propto y$ or $y \propto x$)

Then, in the case of $X = \mathbb{N}$, the usual ordering $\leq$ is an almost complete ordering on $\mathbb{N}$, and likewise $>$, $\geq$ and $<$. More generally, we have: If $\propto$ is an almost complete ordering on X, then so are $\not\propto$, $\overline{\propto}$ and $\cdot\propto$, where

$$\not\propto \ := \{\, (x,y) \in X \times X \,|\, (x,y) \notin \propto \}, \quad \overline{\propto} := \{\, (x,y) \in X \times X \,|\, (y,x) \in \propto \}$$

and

$$\cdot\propto \ := \ \overline{\not\propto} \quad .$$

Let $(W, \cdot)$ be a free monoid over the alphabet X. The unit element of $(W, \cdot)$ is denoted by $\imath$, and the elements of

$$(W, \cdot)^{\propto} := \left\{ w_1 \cdot \ \ldots \ \cdot w_n \in W \,\middle|\, w_1 \propto w_2 \propto \cdots \propto w_n \right\}$$

are called $\propto$-*monotonous*. For any word $w = w_1 \cdot \ldots \cdot w_n \in W$ of *length* $|w| := n$, the *content* of w is defined by

$$c(w) : X \longrightarrow \mathbb{N}_0, \, x \longmapsto |\{\, i \in \underline{n} \,|\, w_i = x \,\}| \quad .$$

Furthermore, for $u = u_1 \cdot \ldots \cdot u_m, v = v_1 \cdot \ldots \cdot v_n \in W$, we write

$$u \propto^* v \quad ,$$

if $u_i \propto v_j$ for all $i \in \underline{m}$, $j \in \underline{n}$.

As is convenient for our purposes, *tableaux* over X are viewed as elements of a free monoid $(T, \bullet)$ over the alphabet W. The unit element of T is denoted by $\ddot{\imath}$. Let $t = t_l \bullet \cdots \bullet t_2 \bullet t_1 \in T$ be a tableau. Then, for all $i \in \underline{l}$, the word t_i is called the *i-th row* of t. Here, the reverse labelling is used for technical reasons. The *shape* of t is defined by

$$\mathrm{sh}(t) : \mathbb{N} \longrightarrow \mathbb{N}_0, \, i \longmapsto \begin{cases} |t_i| & , \quad i \leq l \\ 0 & , \quad i > l \end{cases} \quad ,$$

and $c(t) := c(t_1 \cdot \ldots \cdot t_l)$ is called the *content* of t. The set of all tableaux (of shape r, with content p, resp.), whose rows are $\propto$-monotonous, is denoted by $\overset{\propto}{T}$ ($\overset{\propto}{T}{}^r$, $\overset{\propto}{T}_p$, resp.). Furthermore, we put

$$t_{>z} := t_l \bullet \cdots \bullet t_{z+1} \quad \text{and} \quad t_{<z} := t_{z-1} \bullet \cdots \bullet t_1$$

for all $z \in \underline{l+1} \cup \{0\}$. Any tableau may be visualized by listing its rows one above the other, with the first row on top. In the case of $X = \mathbb{N}$, we have, for example

$$t = (1 \cdot 2 \cdot 4) \bullet (2 \cdot 2) \bullet (3 \cdot 2 \cdot 2) \bullet (1 \cdot 1 \cdot 5 \cdot 4) \quad \sim \quad \begin{array}{cccc} 1 & 1 & 5 & 4 \\ 3 & 2 & 2 & \\ 2 & 2 & & \\ 1 & 2 & 4 & \end{array} \qquad .$$

This tableau t has shape $(4, 3, 2, 3, 0, 0, \ldots)$ and content $(3, 5, 1, 2, 1, 0, 0, \ldots)$.

Now, we define mappings

$$\boxplus : (W, \cdot)^{\propto} \times X \longrightarrow W \quad \text{and} \quad \measuredangle : (W, \cdot)^{\propto} \times X \longrightarrow X$$

as follows: Let $w \in (W, \cdot)^{\propto}$ and $x \in X$. In the case of $w \propto^* x$, we put $w \boxplus x := w$ and $w \measuredangle x := x$. Otherwise, we can find $w^{(1)}, w^{(2)} \in W$, $y \in X$ such that $w = w^{(1)} \cdot y \cdot w^{(2)}$, $w^{(1)} \propto^* x$ and $y \not\propto x$ and put

$$w \boxplus x := w^{(1)} \cdot x \cdot w^{(2)} \quad \text{and} \quad w \measuredangle x := y.$$

In this second case, we say that x *enters* w *in* $(W, \cdot)^{\propto}$, while otherwise, we say that x *passes* w. These definitions of $\boxplus$ and $\measuredangle$ may be extended naturally to $\overset{\propto}{T} \times X$ by induction, namely by putting

$$t \boxplus x := \Big(t_{>1} \boxplus (t_1 \measuredangle x)\Big) \bullet \Big(t_1 \boxplus x\Big) \quad \text{and} \quad t \measuredangle x := t_{>1} \measuredangle (t_1 \measuredangle x).$$

for all $t = t_l \bullet \cdots \bullet t_1 \in \overset{\propto}{T}$ such that $l > 1$. Furthermore, let $\ddot{\imath} \boxplus x := \ddot{\imath}$ and $\ddot{\imath} \measuredangle x := x$. We say that x *enters* t (in $(W, \cdot)^{\propto}$ and $(T, \bullet)$), if $t_{<i} \measuredangle x$ enters t_i in $(W, \cdot)^{\propto}$ for all $i \in \underline{l}$. Otherwise, we say that x *passes* t. Then, particularly, any $x \in X$ enters the empty tableau $\ddot{\imath}$. For example, in the case of $X = \mathbb{N}$ and $\propto = \leq$, we have

$$(2 \cdot 3) \bullet (1 \cdot 1 \cdot 2) \boxplus 1 = (2 \cdot 3 \boxplus 2) \bullet (1 \cdot 1 \cdot 1) = (2 \cdot 2) \bullet (1 \cdot 1 \cdot 1) \quad .$$

Hence 1 enters $(2 \cdot 3) \bullet (1 \cdot 1 \cdot 2)$ in $(W, \cdot)^{\propto}$ and $(T, \bullet)$, and

$$(2 \cdot 3) \bullet (1 \cdot 1 \cdot 2) \measuredangle 1 = 2 \cdot 3 \measuredangle 2 = 3 \quad .$$

Note that, for all $x \in X$ and $w \in (W, \cdot)^{\propto}$ such that x enters w, we have

$$(w \measuredangle x) \;\not\propto\; x \,. \tag{3}$$

Furthermore, for $t^{(1)}, t^{(2)} \in \overset{\propto}{T}$, a simple induction on $|t^{(2)}|$ shows that

$$(t^{(1)} \bullet t^{(2)}) \boxplus x = \Big(t^{(1)} \boxplus (t^{(2)} \measuredangle x)\Big) \bullet \Big(t^{(2)} \boxplus x\Big) \tag{4}$$

and

$$(t^{(1)} \bullet t^{(2)}) \measuredangle x = t^{(1)} \measuredangle (t^{(2)} \measuredangle x). \tag{5}$$

The following observation is of crucial importance in our context: The definitions of $\boxplus$ and $\measuredangle$ do not only depend on the relation $\propto$, but additionally on the products $\cdot$ and $\bullet$ of the underlying monoids $(W, \cdot)$ over X and $(T, \bullet)$ over W. Based on the *inverse product* on W, defined by

$$u \mathbin{\bar{\cdot}} v := v \cdot u$$

for all $u, v \in W$, we obtain again a free monoid $(W, \bar{\cdot})$ over X. Analogously, we may consider the inverse product $\bar{\bullet}$ instead of $\bullet$ on T. For any $w = w_1 \cdot \ldots \cdot w_n \in (W, \cdot)^{\propto}$, we now have $w = w_n \mathbin{\bar{\cdot}} w_{n-1} \mathbin{\bar{\cdot}} \ldots \mathbin{\bar{\cdot}} w_1$ and $w_n \overline{\propto} w_{n-1} \overline{\propto} \cdots \overline{\propto} w_1$. This implies that

$$(W, \cdot)^{\propto} = (W, \bar{\cdot})^{\overline{\propto}} \quad . \tag{6}$$

We shall simply write $W^{\propto}$ for this set. In addition to the mappings $\boxplus$ and $\measuredangle$ based on $\propto$ and the products $\cdot$ on W and $\bullet$ on T, we shall also be interested in the mappings of the same kind arising from the relation $\overline{\propto}$ and the products $\bar{\cdot}$ and $\bar{\bullet}$ instead. Note that, by (6), all these mappings are defined on the same domain $\overset{\propto}{T} \times X$. In order to distinguish, we write

$$x \boxminus t \quad \text{and} \quad x \sphericalangle t \qquad (x \in X, \quad t \in \overset{\propto}{T}) \quad ,$$

if we refer to the mappings based on $\overline{\propto}$, $\bar{\cdot}$ and $\bar{\bullet}$. Furthermore, we will use the following convention: If $x \in X$ enters $t \in \overset{\propto}{T}$ in $(W, \cdot)^{\propto}$ and $(T, \bullet)$, we simply say that x *enters* t. In the case that x enters t in $(W, \bar{\cdot})^{\overline{\propto}}$ and $(T, \bar{\bullet})$, we say that x *inversely enters* t.

Due to this observation, in Proposition 1 and Proposition 2 (and hence, in Lemma 1 and Lemma 2) it suffices to proof the first part. In each case, the second part may then essentially be obtained by *applying* the first part to the inverse ordering and the inverse products. Similarly, in the proofs of the basic tools Proposition 3 and Lemma 4 of the Sections 2 and 3, we may restrict ourselves to considering one implication of the claimed equivalences.

Proposition 1. *Let $t \in \overset{\propto}{T}$ and $x \in X$.*

(a) *We have $t \boxplus x \in \overset{\propto}{T}$. If x enters t, then $t \measuredangle x$ inversely enters $t \boxplus x$, and*

$$(t \measuredangle x) \boxminus (t \boxplus x) = t \quad \textit{and} \quad (t \measuredangle x) \sphericalangle (t \boxplus x) = x\,.$$

(b) *We have* $x \boxminus t \in \overset{\propto}{T}$. *If* x *inversely enters* t, *then* $x \trianglerighteq t$ *enters* $x \boxminus t$, *and*

$$(x \boxminus t) \boxplus (x \trianglerighteq t) = t \quad \text{and} \quad (x \boxminus t) \underline{\angle} (x \trianglerighteq t) = x.$$

Proofs. ad (a): Let $t = t_l \bullet \cdots \bullet t_1$ and assume first that $l = 1$, that is, $w := t \in W^{\propto}$. In the case of $w \propto^* x$ we have $w \boxplus x = w \in W^{\propto}$. Now assume that x enters w. Then there exist $w^{(1)}, w^{(2)} \in W$ and $y \in X$ such that $w = w^{(1)} \cdot y \cdot w^{(2)}$, $w^{(1)} \propto^* x$ and $y \not\propto x$, and we have $w \boxplus x = w^{(1)} \cdot x \cdot w^{(2)}$ and $w \underline{\angle} x = y$. In particular, it follows that $x = y$ or $x \propto y$ and hence $x \propto^* w^{(2)}$. This shows $w \boxplus x \in W^{\propto}$. Furthermore, as $w^{(2)} \overline{\propto}^* y$, $x \not\overline{\propto} y$ and $w \boxplus x = w^{(2)} \bar{\cdot} x \bar{\cdot} w^{(1)}$, we may conclude that y inversely enters $w \boxplus x$ and that

$$y \boxminus (w^{(2)} \bar{\cdot} x \bar{\cdot} w^{(1)}) = w^{(2)} \bar{\cdot} y \bar{\cdot} w^{(1)} = w \quad \text{and} \quad y \trianglerighteq (w^{(2)} \bar{\cdot} x \bar{\cdot} w^{(1)}) = x.$$

This completes the proof for $l = 1$. Now, for $l > 1$, the assertion easily follows by induction using (4) and (5).

ad (b): In any monoid, the inverse product of the inverse product is the initial product again. Hence (b) follows from (a), applied to $(W, \bar{\cdot})$, $(T, \bar{\bullet})$ and $\overline{\propto}$.

$\boxed{QED}$

Our next aim is to introduce the notion of a shifted standard tableau. Let $\mathcal{W}$ be a free monoid over the alphabet $\mathbb{N}$. For any $q = q_1 \cdots q_k \in \mathcal{W}$, we define

$$q_\infty \colon \mathbb{N} \longrightarrow \mathbb{N}_0,\; i \longmapsto \begin{cases} q_i & , \quad i \le k \\ 0 & , \quad i > k \end{cases}$$

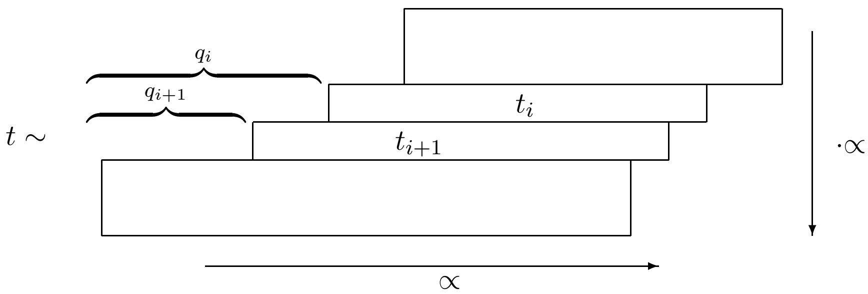
and write q instead q_∞ whenever confusion is impossible, for instance for the shape of a tableau. The word q is called a *partition*, if $q_1 \ge q_2 \ge \cdots \ge q_k$. If, additionally, $q_1 + \cdots + q_k = n$, we say that q is a partition of n ($q \vdash n$). For all $u = u_1 \cdot \ldots \cdot u_n, v = v_1 \cdot \ldots \cdot v_k \in W$ and $j \in \mathbb{N}_0$ we write

$$u \propto_j v \quad ,$$

if $|u| + j \ge |v|$ and $u_\nu \propto v_{j+\nu}$ for all $\nu \in \underline{n} \cap \underline{k-j}$. Let $q = q_1 \cdots q_k \in \mathcal{W}$ be a partition. We put $d_i := (q_\infty)_i - (q_\infty)_{i+1}$ for all $i \in \mathbb{N}$ and

$$S^q\overset{\propto}{T} := \{\, t = t_l \bullet \cdots \bullet t_1 \in \overset{\propto}{T} \mid t_i \cdot\!\propto_{d_i} t_{i+1} \text{ for all } i \in \underline{l-1} \},$$

The elements of $S^q\overset{\propto}{T}$ are called *q-shifted standard tableaux* (with respect to $\propto$). For all $r \colon \mathbb{N} \longrightarrow \mathbb{N}_0$, $p \colon X \longrightarrow \mathbb{N}_0$ we put $S^q\overset{\propto}{T}{}^r := S^q\overset{\propto}{T} \cap \overset{\propto}{T}{}^r$ and $S^q\overset{\propto}{T}{}^r_p := S^q\overset{\propto}{T}{}^r \cap \overset{\propto}{T}_p$. In the case that q is the empty partition the upper index q is omitted. Any q-shifted standard tableau t may be visualized by shifting the i-th row of t q_i positions to the right, for all i. Then, in this visualization, each row is $\propto$-monotonous, while each column is monotonous with respect to $\cdot\!\propto$:



Note that, for all $s \in \mathcal{W}$ and any partition $q \in \mathcal{W}$ such that $S^q\overset{\propto}{T}{}^s \neq \emptyset$, we have

$$(q_\infty)_i + (s_\infty)_i \geq (q_\infty)_{i+1} + (s_\infty)_{i+1} \qquad \text{for all } i \in \mathbb{N}. \tag{7}$$

In particular, $\mathrm{sh}(t)$ is non-increasing for all $t \in S\overset{\propto}{T}$.

Proposition 2. *Let $t = t_l \bullet \cdots \bullet t_1 \in \overset{\propto}{T}$ and $x \in X$ and assume that $l \geq 2$.*

(a) *Let $q \in \mathcal{W}$ be a partition such that $t \in S^q\overset{\propto}{T}$.*
If x enters $t_{<l}$, then

$$t \boxplus x \in S^q\overset{\propto}{T}.$$

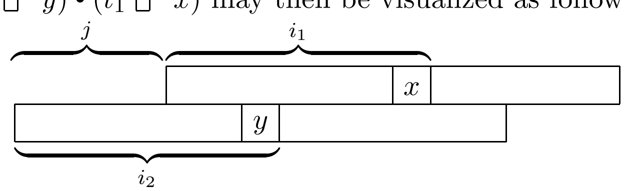
If x enters $t_{<l}$, but passes t, then

$$\Big(t_l \cdot (t_{<l} \measuredangle x)\Big) \bullet (t_{<l} \boxplus x) \in S^q\overset{\propto}{T}.$$

(b) *If $(t_l \cdot x) \bullet t_{<l} \in S\overset{\propto}{T}$, then x inversely enters $t_{<l}$, and*

$$t_l \bullet (x \boxplus t_{<l}) \in S\overset{\propto}{T}.$$

Proofs. ad (a): By Proposition 1(a), we have $t \boxplus x \in \overset{\propto}{T}$. Assume first that $l = 2$. Let $t_2 = v_1 \cdot \ldots \cdot v_k$ and put $j := q_1 - q_2$. As x enters t_1, there exist $u^{(1)}, u^{(2)} \in W$ and $y \in X$ such that $t_1 = u^{(1)} \cdot y \cdot u^{(2)}$, $t_1 \boxplus x = u^{(1)} \cdot x \cdot u^{(2)}$ and $t_1 \measuredangle x = y$. We put $i_1 := |u^{(1)}| + 1$. In the case of $j + i_1 > |t_2|$ both assertions follow easily. Let $j + i_1 \leq |t_2|$. As $t_1 \cdot\!\propto_j t_2$, we have $v_{j+i_1} \not\propto y$. Hence y enters t_2 and x enters t. It remains to be shown that $t_1 \boxplus x \cdot\!\propto_j t_2 \boxplus y$. We choose $v^{(1)}, v^{(2)} \in W$ such that $t_2 \boxplus y = v^{(1)} \cdot y \cdot v^{(2)}$ and put $i_2 := |v^{(1)}| + 1 \leq j + i_1$. The tableau $(t_2 \boxplus y) \bullet (t_1 \boxplus x)$ may then be visualized as follows:



In the case of $i_2 = j + i_1$ it follows that $t_1 \boxplus x \cdot\!\propto_j t_2 \boxplus y$, as $t_1 \cdot\!\propto_j t_2$ and $y \not\propto x$. Let $i_2 < j + i_1$. Then we have

$$(t_2 \boxplus y)_{j+i_1} = (t_2)_{j+i_1} \not\propto y \not\propto x = (t_1 \boxplus x)_{i_1}.$$

If $i_2 > j$, we may conclude that, additionally,

$$(t_2 \boxplus y)_{i_2} = y \not\propto x \quad \text{and} \quad (t_1 \boxplus x)_{i_2-j} = u^{(1)}{}_{i_2-j} \propto x,$$

hence $(t_2 \boxplus y)_{j+(i_2-j)} \not\propto (t_1 \boxplus x)_{i_2-j}$. This shows $t_1 \boxplus x \cdot\!\propto_j t_2 \boxplus y$ and completes the proof for $l = 2$. For $l > 2$, we can use (4) and proceed by an easy induction.

ad (b): Let $m := |t_l|+1$ and $\tilde{t} := (t_l \cdot x) \bullet t_{<l} \in S\overset{\propto}{T}$. Then $\mathrm{sh}(\tilde{t})$ is non-increasing. This implies that $|t_{l-1}| \geq m$ and $x \not\propto (t_{l-1})_m$. Thus x inversely enters t_{l-1} and, by a simple induction, also $t_{<l}$. More precisely, we can find $v^{(1)}, v^{(2)} \in W$ such that $x \boxplus t_{l-1} = v^{(1)} \cdot x \cdot v^{(2)}$ and $|v^{(1)}| \geq |t_l|$. Let $s := \mathrm{sh}(t_{<l})$ and $q \in \mathcal{W}$ be the unique partition such that $(q_\infty)_i = s_1 - s_{l-i}$ for all $i \in \underline{l-1}\lrcorner$ and $(q_\infty)_i = 0$ for all $i \geq l$. Then we have

$$t_{<l} = t_1 \,\bar{\bullet}\, \cdots \,\bar{\bullet}\, t_{l-1} \in S^q\overline{\overset{\propto}{T}}$$

with respect to the inverse products on W and T (for more details, see Lemma 6). Applying (a), we have $x \boxplus t_{<l} \in S^q\overline{\overset{\propto}{T}}$ with respect to the inverse products on W and T. But this is equivalent to $x \boxplus t_{<l} \in S\overset{\propto}{T}$ with respect to the initial products on W and T, and the proof is completed.

$\boxed{QED}$

Let $t = t_l \bullet \cdots \bullet t_1 \in \overset{\propto}{T}$ and $x \in X$. We choose $z \in \underline{l+1}\lrcorner$ maximal such that x enters $t_{<z}$ and define

$$\mathrm{z}(x,t) := z$$

and

$$t \boxtimes\!\boxplus x := t_{>z} \bullet \Big(t_z \cdot (t_{<z} \measuredangle x)\Big) \bullet (t_{<z} \boxplus x)\,,$$

where, in the case of $z = l + 1$, we put $t_{l+1} := \iota$.

Now, on the other hand, if $z \in \underline{l}\lrcorner$ such that $m := |t_z| > 0$, we denote by $y := (t_z)_m$ the final letter of t_z and define

$$\mathrm{b}(z,t) := y \searrow t_{<z}$$

and

$$t[z] := \begin{cases} t_{>z} \bullet (t_{z,1} \cdots t_{z,m-1}) \bullet (y \boxplus t_{<z}) & , \quad m > 1 \\ t_{>z} \bullet (y \boxplus t_{<z}) & , \quad m = 1 \end{cases} .$$

We observe that

$$\mathrm{sh}(t \boxtimes x)_i = \begin{cases} \mathrm{sh}(t)_i + 1 & , \quad i = \mathrm{z}(x,t) \\ \mathrm{sh}(t)_i & , \quad \text{otherwise} \end{cases} \tag{8}$$

and, in the case that $\mathrm{sh}(t)$ is non-increasing and $\mathrm{sh}(t)_z > \mathrm{sh}(t)_{z+1}$,

$$\mathrm{sh}(t[z])_i = \begin{cases} \mathrm{sh}(t)_i - 1 & , \quad i = z \\ \mathrm{sh}(t)_i & , \quad \text{otherwise} \end{cases} \tag{9}$$

for all $i \in \mathbb{N}$.

Lemma 1. *Let* $t = t_l \bullet \cdots \bullet t_1 \in \overset{\propto}{T}$.

(a) *For all* $x \in X$, *we have*

$$\Big(t \boxtimes x\Big)[\mathrm{z}(x,t)] = t \quad \text{and} \quad \mathrm{b}\Big(\mathrm{z}(x,t), t \boxtimes x\Big) = x.$$

(b) *If* $t \in S\overset{\propto}{T}$, *for all* $z \in \underline{l}$ *such that* $sh(t)_z > sh(t)_{z+1}$, *we have*

$$t[z] \boxtimes \mathrm{b}(z,t) = t \quad \text{and} \quad \mathrm{z}\Big(\mathrm{b}(z,t), t[z]\Big) = z.$$

Proofs. ad (a): Proposition 1(a)

ad (b): Let $t \in S\overset{\propto}{T}$ and $z \in \underline{l}$ such that $\mathrm{sh}(t)_z > \mathrm{sh}(t)_{z+1}$. Let $\tilde{t} := t[z]$. Then, by Proposition 2(b), $x := (t_z)_{|t_z|}$ inversely enters $t_{<z}$. Hence, by Proposition 1(b), $\mathrm{b}(z,t) = x \gtrdot t_{<z}$ enters $\tilde{t}_{<z} = x \boxplus t_{<z}$ and

$$\tilde{t}_z \ \propto^* \ x \ = \ (x \boxplus t_{<z}) \measuredangle (x \gtrdot t_{<z}) \ = \ t[z]_{<z} \measuredangle \mathrm{b}(x, \tilde{t}) \quad .$$

This implies that $\mathrm{b}(z,t)$ passes $\tilde{t}_{<z+1}$ and hence $\mathrm{z}(\mathrm{b}(z,t), \tilde{t}) = z$. Furthermore, we can conclude that $\tilde{t} \boxtimes \mathrm{b}(z,t) = t$ from Proposition 1(b).

$\boxed{QED}$

As an immediate consequence of Proposition 2 we observe:

Lemma 2. *Let* $t = t_l \bullet \cdots \bullet t_1 \in S\overset{\propto}{T}$.

(a) *For all* $x \in X$, *we have* $t \boxtimes x \in S\overset{\propto}{T}$.

(b) *For all* $z \in \underline{l}$ *such that* $sh(t)_z > sh(t)_{z+1}$, *we have* $t[z] \in S\overset{\propto}{T}$.

Now we define mappings

$$P_\propto\colon W \longrightarrow T \quad \text{ and } \quad Q_\propto\colon W \longrightarrow \mathcal{W}$$

as follows: First, we put $P_\propto(\imath) := \ddot{\imath}$ and $Q_\propto(\imath) := \imath$. Let $w = w_1 \cdot \ldots \cdot w_n \in W\backslash\{\imath\}$ and $\tilde{w} := w_1 \cdot \ldots \cdot w_{n-1}$. Inductively, we put $P_\propto(w) := P_\propto(\tilde{w}) \boxtimes w_n$ and $Q_\propto(w) := Q_\propto(\tilde{w})\,\mathrm{z}(w_n, P_\propto(\tilde{w}))$. The definition of the mappings $P_\propto$ and $Q_\propto$ is essentially due to Schensted [13]. The tableau $P_\propto(w)$ is often referred to as *P-symbol*, while the word $Q_\propto(w)$ is called *Q-symbol* of w (with respect to $\propto$).

Any word $p = p_1 \cdots p_n \in \mathcal{W}$ is called *standard* (or *lattice permutation*), if $c(p_1 \cdots p_i)$ is non-increasing for all $i \in \underline{n}$. For example, the word 1112213 is standard, while the word 1122213 is not. The set of all lattice permutations in $\mathcal{W}$ is denoted by $\mathcal{L}$. Furthermore, for all $r \in \mathcal{W}$, we denote by $\mathcal{L}_r$ the set of all lattice permutation with content r_∞.

Theorem 1. *Let $S\overset{\propto}{T} \overset{=}{\times} \mathcal{L}$ be the set of all pairs $(t_l \bullet \cdots \bullet t_1, p) \in S\overset{\propto}{T} \times \mathcal{L}$ such that $sh(t) = c(p)$ and $t_i \neq \imath$ for all $i \in \underline{l}$. Then the mapping*

$$W \longrightarrow S\overset{\propto}{T} \overset{=}{\times} \mathcal{L},\ w \longmapsto \Big(P_\propto(w), Q_\propto(w)\Big) \tag{10}$$

is a bijection, and $c(P_\propto(w)) = c(w)$ for all $w \in W$.

For example, in the case of $X = \mathbb{N}$ and

$$4 \propto 3 \propto 2 \propto 1,\ 4 \propto 4,\ 3 \propto 3,\ 2 \not\propto 2 \quad \text{ and } \quad 1 \propto 1 \quad , \tag{11}$$

for the word

$$w = 3 \cdot 3 \cdot 2 \cdot 4 \cdot 2 \cdot 1 \cdot 4 \quad , \tag{12}$$

we obtain

$$P_\propto(w) \sim \begin{array}{llll} 4 & 4 & 2 & 1 \\ 3 & 3 & & \\ 2 & & & \end{array} \in S\overset{\propto}{T}^{421} \quad \text{ and } \quad Q_\propto(w) = 1112213 \in \mathcal{L}_{421}. \tag{13}$$

Proof of the theorem. The second assertion concerning the content can be shown easily by induction. We put $\mathcal{A} := S\overset{\propto}{T} \overset{=}{\times} \mathcal{L}$ and define $\alpha\colon X \times \mathcal{A} \longrightarrow \mathcal{A}\backslash\{(\ddot{\imath}, \imath)\}$ by

$$\Big(x,\ (t,p)\Big) \longmapsto \Big(t \boxtimes x,\ p\,\mathrm{z}(x,t)\Big).$$

By Lemma 2(a) and (8), we have indeed $(X \times \mathcal{A})\alpha \subseteq \mathcal{A}\backslash\{(\ddot{\imath}, \imath)\}$. Furthermore, for the mapping

$$\beta\colon \mathcal{A}\backslash\{(\ddot{\imath}, \imath)\} \longrightarrow X \times \mathcal{A},\ (s, p_1 \cdots p_n) \longmapsto \Big(\mathrm{b}(p_n, s),\ (s[p_n],\ p_1 \cdots p_{n-1})\Big),$$

we have $(s,p)\beta \in X \times \mathcal{A}$ for all (s,p) by Lemma 2(b) and (9). Applying Lemma 1(a), we obtain $\alpha\beta = \mathrm{id}_{X\times\mathcal{A}}$, while Lemma 1(b) implies that $\beta\alpha = \mathrm{id}_{\mathcal{A}\setminus\{(\dot{\imath},\dot{\imath})\}}$. Hence α is a bijection. Now, for all $n \in \mathbb{N}_0$, we put

$$W^n := \{\, w \in W \mid |w| = n \,\}$$

and define

$$\gamma_n \colon W^n \longrightarrow \bigcup_{r \vdash n} S\overset{\propto}{\mathcal{T}}{}^r \times \mathcal{L}_r,\; w \longmapsto \Big(P_\propto(w), Q_\propto(w)\Big).$$

Then $w\gamma_n = (w_n, (w_1 \cdot \ldots \cdot w_{n-1})\gamma_{n-1})\alpha$ for all $w = w_1 \cdot \ldots \cdot w_n \in W$, and the proof may be completed easily by induction. $\boxed{QED}$

Let $\mathcal{T}$ be a set of tableaux over the alphabet $\mathbb{N}$, that is, a free monoid over $\mathcal{W}$. Let $t \in \mathcal{T}$ and assume that there exists an $n \in \mathbb{N}$ such that

$$c(t)_i = \begin{cases} 1 & , \quad i \le n \\ 0 & , \quad i > n \end{cases}$$

for all $i \in \mathbb{N}$, that is, any letter $i \in \underline{n}$ occurs exactly once in t. Then t is called a *Young tableau* (over $\mathbb{N}$) or is said to be of *Young type*. The set of all Young tableaux is denoted by $Y\mathcal{T}$. Furthermore, we put

$$SY\mathcal{T} := S\overset{\le}{\mathcal{T}} \cap Y\mathcal{T}.$$

For all $i \in \underline{n}$, let z_i be the number of the row of t containing i. Then

$$t_{\mathfrak{z}} := z_1 \cdots z_n \in \mathcal{W}$$

is called the *row word* of t. For all $r \colon \mathbb{N} \longrightarrow \mathbb{N}_0$, denoting by $\mathcal{W}_r$ the set of words $w \in \mathcal{W}$ with content r, we obtain a bijection

$$Y\mathcal{T}^r \longrightarrow \mathcal{W}_r,\; t \longmapsto t_{\mathfrak{z}} \quad . \tag{14}$$

Furthermore, we have $t \in SY\mathcal{T}$ if and only if $t_{\mathfrak{z}} \in \mathcal{L}$. This observation is due to Macmahon (96 in [10]). For the special choice of $X = \mathbb{N}$ and $\propto = \le$, we thus obtain the following classical correspondence due to Robinson [12] and Schensted [13].

Corollary 1. *Let $r \colon \mathbb{N} \longrightarrow \mathbb{N}_0$. Then the mapping*

$$w \longmapsto \Big(P_\le(w), (Q_\le(w))_{\mathfrak{z}}{}^{-1}\Big)$$

is a bijection from the set of words over $\mathbb{N}$ of content r onto the set of pairs of standard tableaux over $\mathbb{N}$ of the same shape, the first of which has content r and the second of which is of Young type.

2 Descent Sets

For all $w = w_1 \cdot \ldots \cdot w_n \in W$, the *descent set* of w (with respect to $\propto$) is defined by

$$\mathcal{D}_\propto(w) := \{\, i \in \underline{n-1} \,|\, w_i \not\propto w_{i+1} \,\} \quad . \tag{15}$$

For example, the descent set of the word w defined in (12) (with respect to $\propto$ defined in (11)) is $\{3, 6\}$. In this section, we will show that the descent set of any word w (with respect to $\propto$) is equal to the descent set of its $Q_\propto$-symbol (with respect to $\geq$). In the above example, bearing in mind (13), we obtain indeed $\mathcal{D}_\geq(Q_\propto(w)) = \mathcal{D}_\geq(1112213) = \{3, 6\}$. In the case of $X = \mathbb{N}$ and $\propto = \leq$, this result is due to Schützenberger [14, Remarque 2], and, independently, to Foulkes [3, Theorem 2.1]. In the case of $\propto = <$, we obtain Knuth's result on the so-called dual correspondence [7, Theorem 1*]. It is interesting to compare the proofs of the Theorems 1 and 1* in [7] with those of Proposition 3 and Lemma 3 below.

Proposition 3. *Let $w \in W^\propto$ and $x, y \in X$. Assume that x enters w and y enters $w \boxplus x$. Then the following conditions are equivalent:*

(i) $x \propto y$,

(ii) $(w \measuredangle x) \propto \Big((w \boxplus x) \measuredangle y\Big)$.

PROOF. We choose $w^{(1)}, w^{(2)} \in W$ and $a \in X$ such that $w = w^{(1)} \cdot a \cdot w^{(2)}$, $w \boxplus x = w^{(1)} \cdot x \cdot w^{(2)}$ and $w \measuredangle x = a$. Then, assuming (i), we have $w^{(1)} \cdot x \propto^* y$ and hence $a \propto ((w \boxplus x) \measuredangle y)$, as $a \propto^* w^{(2)}$. This shows (ii). Now, putting $\tilde{x} := (w \boxplus x) \measuredangle y$, $\tilde{y} := w \measuredangle x$ and $\tilde{w} := (w \boxplus x) \boxplus y$, we have

$$y = \tilde{x} \gtrdot \tilde{w} \quad \text{and} \quad x = \tilde{y} \gtrdot (\tilde{x} \boxminus \tilde{w}) \quad ,$$

by Proposition 1(a). The remaining implication is thus an application of the one already proved, applied to $\tilde{x}$, $\tilde{y}$, $\tilde{w}$, $\overline{\propto}$ and $\boxminus$, $\gtrdot$ instead of x, y, w, $\propto$ and $\boxplus$, $\measuredangle$. $\boxed{QED}$

For all $t = t_l \bullet \cdots \bullet t_1 \in \overset{\propto}{T}$ and $x \in X$, we define

$$\mathrm{s}(x, t) := |(t \boxtimes x)_{\mathrm{z}(x,t)}|.$$

and observe that

$$\mathrm{z}(x, t) = \mathrm{z}(t_1 \measuredangle x, t_{>1}) + 1 \quad \text{and} \quad \mathrm{s}(x, t) = \mathrm{s}(t_1 \measuredangle x, t_{>1}) \tag{16}$$

for all $t = t_l \bullet \cdots \bullet t_1 \in \overset{\propto}{T} \backslash \{\ddot{\imath}\}$ and $x \in X$ such that x enters t_1.

Lemma 3. *For all $t \in \overset{\propto}{T}$ and $x, y \in X$, the following conditions are equivalent:*

(i) $x \propto y$,

(ii) $\mathrm{z}(x,t) \geq \mathrm{z}(y, t \boxtimes x)$.

If, additionally, $t \in S\overset{\propto}{T}$, we obtain the following third equivalent condition:

(iii) $\mathrm{s}(x,t) < \mathrm{s}(y, t \boxtimes x)$.

Proof. Let $t = t_l \bullet \cdots \bullet t_1 \in \overset{\propto}{T}$, $x, y \in X$, $z_x := \mathrm{z}(x,t)$, $z_y := \mathrm{z}(y, t \boxtimes x)$ and $s_x := \mathrm{s}(x,t)$, $s_y := \mathrm{s}(y, t \boxtimes x)$. In the case of $t \in S\overset{\propto}{T}$, we have $s_y \leq |t_1| + 2$, as $\mathrm{sh}(t)$ is non-increasing. Now assume first that x passes t_1 or $t = \ddot{\imath}$. Then we have $z_x = 1$ and $s_x = |t_1| + 1$. Putting $t_1 := i$ in the case of $t = \ddot{\imath}$, we obtain

$$x \propto y \iff t_1 \cdot x \propto^* y \iff z_y = 1 \iff z_x \geq z_y$$

and, in the case of $t \in S\overset{\propto}{T}$, $x \propto y \iff s_y = |t_1| + 2 \iff s_x < s_y$ as asserted. Now assume that $t \neq \ddot{\imath}$ and that x enters t_1. Then there exist $\tilde{x} \in X$ and $u^{(1)}, u^{(2)} \in W$ such that $t_1 = u^{(1)} \cdot \tilde{x} \cdot u^{(2)}$ and $t_1 \boxplus x = u^{(1)} \cdot x \cdot u^{(2)}$. If $(t_1 \boxplus x) \propto^* y$, we may conclude that $x \propto y$, $z_x > 1 = z_y$ and $s_x \leq |t_1| < s_y$ in the case of $t \in S\overset{\propto}{T}$, by Lemma 2(a). If, on the other hand, y enters $t_1 \boxplus x$, we put $\tilde{y} := (t_1 \boxplus x) \nleq y$ and obtain inductively

$$x \propto y \iff \tilde{x} \propto \tilde{y} \iff \mathrm{z}(\tilde{x}, t_{>1}) \geq \mathrm{z}(\tilde{y}, t_{>1} \boxtimes \tilde{x}) \iff z_x \geq z_y$$

and the corresponding equivalence for s, by Proposition 3 and (16). $\boxed{QED}$

The mappings $\nleq$, $\boxplus$, $\boxtimes$, z and s are defined on the set $\overset{\propto}{T} \times X$. In order to simplify our notations in the sequel, they will be extended to $\overset{\propto}{T} \times W$ canonically as follows: Let $t \in \overset{\propto}{T}$. First, we put $t \nleq i := i$, $t \boxplus i := t$ and let $\mathrm{z}(i,t)$ be the unit element of $\mathcal{W}$. Now let $w = w_1 \cdot \ldots \cdot w_n \in W \backslash \{i\}$ and $\tilde{w} := w_1 \cdot \ldots \cdot w_{n-1}$. Then, inductively, we put

$$t \boxplus w := (t \boxplus \tilde{w}) \boxplus w_n, \quad t \nleq w := (t \nleq \tilde{w}) \cdot \Big((t \boxplus \tilde{w}) \nleq w_n\Big)$$

and

$$\mathrm{z}(w,t) := \mathrm{z}(\tilde{w}, t)\, \mathrm{z}(w_n, t \boxtimes \tilde{w}).$$

The mapping $\boxtimes$ (s, resp.) is extended in the same way as $\boxplus$ (z, resp.). Particularly, we then have

$$Q_\propto(w) = \mathrm{z}(w, \ddot{\imath}) \tag{17}$$

for all $w \in W$. For any $t \in \overset{\propto}{T}$ and any $w = w_1 \cdot \ldots \cdot w_n \in W$, we say that w *enters* t, if w_i enters $t \boxplus (w_1 \cdot \ldots \cdot w_{i-1})$ for all $i \in \underline{n}$. Otherwise, we say that w passes t.

Note that, for all $u, v \in W$ and $t \in T$, the inductive definition given above implies that

$$t \boxplus (u \cdot v) = (t \boxplus u) \boxplus v, \quad t \measuredangle (u \cdot v) = (t \measuredangle u) \cdot \Big((t \boxplus u) \measuredangle v\Big) \tag{18}$$

and

$$\mathrm{z}(u \cdot v, t) = \mathrm{z}(u, t)\, \mathrm{z}(v, t \boxtimes u). \tag{19}$$

Analogous identities hold for the mappings $\boxtimes$ and s.

Now, applying Lemma 3 and using induction, we obtain the transfer of descent sets mentioned at the beginning of this Section.

Theorem 2. *For all $w \in W$, $t \in S\overset{\propto}{T}$, we have*

$$\mathcal{D}_\propto(w) = \mathcal{D}_\geq(\mathrm{z}(w, t)) = \mathcal{D}_<(\mathrm{s}(w, t)).$$

In particular, for $t = \ddot{\imath}$, we have $\mathcal{D}_\propto(w) = \mathcal{D}_\geq(Q_\propto(w))$.

3 Shifted fillings

The transfer of the descent set is a special case of a more general property that is invariant under the Q-symbol, as will be shown now.

Let $n \in \mathbb{N}$, $w \in W$ such that $|w| = n$ and let $r = r_1 \cdots r_l \in \mathcal{W}$ such that $r_1 + \cdots + r_l = n$. Then there exist unique words $w^{(1)}, \ldots, w^{(l)} \in W$ such that $|w^{(i)}| = r_i$ for all $i \in \underline{l}$ and $w = w^{(l)} \cdot \ldots \cdot w^{(1)}$. We put

$$\mathrm{Tab}_r(w) := w^{(l)} \bullet \cdots \bullet w^{(1)} \in T.$$

Let $q \in \mathcal{W}$ be a partition. If $\mathrm{Tab}_r(w) \in S^q\overset{\propto}{T}{}^r$, then w is called a *q-shifted $\propto$-filling of shape* r. For all $U \subseteq W$, we define

$$S^q\overset{\propto}{U}{}^r := \{\, w \in U \mid |w| = n,\ \mathrm{Tab}_r(w) \in S^q\overset{\propto}{T}{}^r \,\}$$

to be the set of all q-shifted $\propto$-fillings of shape r in U. If q is the empty partition, the upper index q is omitted. In this section, we will particularly show that any word $w \in W$ is a q-shifted $\propto$-filling of shape r if and only if $Q_\propto(w)$ is a q-shifted $\geq$-filling of shape r in $\mathcal{W}$ (Theorem 3).

For example, for the word $w = 3 \cdot 3 \cdot 2 \cdot 4 \cdot 2 \cdot 1 \cdot 4$ considered in (12), we have

$$\mathrm{Tab}_{133}(w) \sim \begin{array}{cccc} & & & 4 \\ & 4 & 2 & 1 \\ 3 & 3 & 2 & \end{array} \in S^{31}\overset{\propto}{\mathcal{T}}^{133},$$

(with $\propto$ defined in (11)), hence $w \in S^{31}\overset{\propto}{W}^{133}$. Recalling (13) and observing that

$$\mathrm{Tab}_{133}(1112213) \sim \begin{array}{cccc} & & & 3 \\ & 2 & 2 & 1 \\ 1 & 1 & 1 & \end{array} \in S^{31}\overset{\geq}{\mathcal{T}}^{133},$$

we obtain that $Q_\propto(w) \in S^{31}\overset{\geq}{\mathcal{W}}^{133}$.

Now, for any $D = \{d_1, \ldots, d_k\} \subseteq \underline{n-1|}$ $(d_1 < \cdots < d_k)$ we may define $q := (d_k - k) \cdots (d_1 - 1)$ and $r = (n - d_k)(d_k - d_{k-1}) \cdots (d_2 - d_1)d_1$ and observe that

$$\mathcal{D}_\propto(w) = D \iff w \in S^q\overset{\propto}{W}^r$$

for all $w \in W$. Thus the transfer of the descent set described in Theorem 2 is indeed a special case of Theorem 3.

Proposition 4. *Let $t = t_l \bullet \cdots \bullet t_1 \in \overset{\propto}{T}$, $w \in W^\propto$ and $v \in W$.*

(a) *If v enters t_1, then*

$$t \boxtimes v = \Big(t_{>1} \boxtimes (t_1 \measuredangle v)\Big) \bullet (t_1 \boxplus v).$$

(b) *If $v = v_1 \cdot \ldots \cdot v_m$ is $\propto$-monotonous, then there exist $w^{(1)}, \ldots, w^{(m+1)} \in W$ such that the first row of $w \boxtimes v$ is given by*

$$(w \boxtimes v)_1 = w^{(1)} \cdot v_1 \cdot w^{(2)} \cdot v_2 \cdot \ldots \cdot w^{(m)} \cdot v_m \cdot w^{(m+1)}.$$

(c) *If v enters w, then*

$$(w \measuredangle v) \boxminus (w \boxplus v) = w \quad \textit{and} \quad (w \measuredangle v) \geqslant (w \boxplus v) = v.$$

Proof. All three claims may be proved easily by induction on $m := |v|$. We restrict ourselves to proving the first part of (c). For $m = 0$, it is simply the definition. Let $m > 0$. We put $u := w \measuredangle v = u_1 \bar{\cdot} \ldots \bar{\cdot} u_m$, $\tilde{u} = u_2 \bar{\cdot} \ldots \bar{\cdot} u_m$ and $\tilde{v} = v_1 \cdot \ldots \cdot v_{m-1}$. Then we have

$$u_m \cdot \ldots \cdot u_1 = w \measuredangle v = (w \measuredangle \tilde{v}) \cdot \Big((w \boxplus \tilde{v}) \measuredangle v_n\Big),$$

hence $u_1 = (w \boxplus \tilde{v}) \measuredangle v_m$ and $\tilde{u} = w \boxplus \tilde{v}$. Applying Proposition 1(a), we obtain

$$u \boxdot (w \boxplus v) = \tilde{u} \boxdot \Big(u_1 \boxdot ((w \boxplus \tilde{v}) \boxplus v_n)\Big) = \tilde{u} \boxdot (w \boxplus \tilde{v}) = w \; .$$

$\boxed{QED}$

Let $j \in \mathbb{N}_0$. For all $w = w_1 \cdot \ldots \cdot w_n \in W$, $t = t_l \bullet \cdots \bullet t_1 \in T$, we put

$$w^{\leq j} := \begin{cases} w_1 \cdot \ldots \cdot w_j & , \quad j < n \\ w & , \quad j \geq n \end{cases} \qquad \text{and} \qquad t^{\leq j} := t_l^{\leq j} \bullet \cdots \bullet t_1^{\leq j}.$$

In the same way, we define $t^{<j}$ and $t^{>j}$.

Lemma 4. *Let $u, v, w \in W^{\propto}$. Assume that $m := |u| = |v|$ and that $u \cdot v$ enters w. Then the following conditions are equivalent:*

(i) $u \cdot v \in S\overset{\propto}{W}^{mm}$,

(ii) $w \measuredangle (u \cdot v) \in S\overset{\propto}{W}^{mm}$.

PROOF. Let $u \cdot v \in S\overset{\propto}{W}^{mm}$. Then, particularly, we have $u, v \in W^{\propto}$ and hence $w \measuredangle u$, $(w \boxplus u) \measuredangle v \in W^{\propto}$, by Proposition 3. Furthermore, by Proposition 4(b), there exist words $w^{(1)}, \ldots, w^{(m+1)} \in W$ such that

$$w \boxplus u = (w \boxtimes u)_1 = w^{(1)} \cdot u_1 \cdot w^{(2)} \cdot u_2 \cdot \ldots \cdot w^{(m)} \cdot u_m \cdot w^{(m+1)}.$$

As $u_i \not\propto v_i$ for all $i \in \underline{m}$, a simple induction shows that, for all $j \in \underline{m}$, there exist $a^{(j)}, b^{(j)} \in W$ such that

$$(*) \quad w \boxplus (u \cdot v^{<j}) = a^{(j)} \cdot v_{j-1} \cdot b^{(j)} \cdot w^{(j)} \cdot u_j \cdot \ldots \cdot w^{(m)} \cdot u_m \cdot w^{(m+1)} \; ,$$

where $a^{(1)} = v_0 = b^{(1)} := i$. Let $j \in \underline{m}$ and put $\tilde{v}_j := (w \boxplus (u \cdot v^{<j})) \measuredangle v_j$. Then, bearing in mind $(*)$, we observe that $u_j = \tilde{v}_j$ or $u_j \not\propto \tilde{v}_j$. On the other hand, we have $\tilde{u}_j := (w \boxplus u^{<j}) \measuredangle u_j \not\propto u_j$, by (3). This implies that $\tilde{u}_j \not\propto \tilde{v}_j$ and thus (ii). The remaining implication can be obtained now by applying the one already proved to $\tilde{u} := w \measuredangle u$, $\tilde{v} := (w \boxplus u) \measuredangle v$, $\tilde{w} := w \boxplus u \cdot v$, $\reflectbox{$\measuredangle$}$ and $\overline{\propto}$ and using Proposition 4(c). $\boxed{QED}$

For all $s = s_k \bullet \cdots \bullet s_1, t = t_l \bullet \cdots \bullet t_1 \in T$, we put

$$s \star t := (s_{\max\{k,l\}} \cdot t_{\max\{k,l\}}) \bullet \cdots \bullet (s_1 \cdot t_1) \in T,$$

where $s_i := i$ for all $i > k$ or, resp., $t_i := i$ for all $i > l$. Note that $t = t^{\leq j} \star t^{>j}$ for all $t \in T$ and $j \in \mathbb{N}_0$.

Proposition 5. *Let $y, z \in X$ and $t = t_l \bullet \cdots \bullet t_1 \in S\overset{\propto}{T}$.*

(a) *If z passes t_1, we have*

$$t \boxtimes z = t \star z \quad \textit{and} \quad \Big(\mathrm{z}(z,t), \mathrm{s}(z,t)\Big) = (1, |t_1|+1).$$

(b) *If $j \in \underline{|t_1|}$ and y enters $t_1^{\leq j}$, we have*

$$t \boxtimes y = (t^{\leq j} \boxtimes y) \star t^{>j} \quad \textit{and} \quad \Big(\mathrm{z}(y,t), \mathrm{s}(y,t)\Big) = \Big(\mathrm{z}(y, t^{\leq j}), \mathrm{s}(y, t^{\leq j})\Big).$$

Proofs. ad (a): Definition.

ad (b): Let $j \in \underline{|t_1|}$ such that $t_1^{\leq j} \not\propto^* y$. Then there exist $u^{(1)}, u^{(2)} \in W$ and $\tilde{y} \in X$ such that $t_1 = u^{(1)} \cdot \tilde{y} \cdot u^{(2)}$, $t_1 \boxplus y = u^{(1)} \cdot y \cdot u^{(2)}$, $\tilde{y} \not\propto y$ and $|u^{(1)}| < j$. We have $|t_2| \leq |u^{(1)}| < j$ or $(t_2)_{|u^{(1)}|+1} \not\propto \tilde{y}$, hence $t_2^{\leq j} \not\propto^* \tilde{y}$. In both cases, putting $\tilde{t} := t_{>1}$, it follows inductively that

$$\begin{aligned} t \boxtimes y &= (\tilde{t} \boxtimes \tilde{y}) \bullet (t_1 \boxplus y) = \Big((\tilde{t}^{\leq j} \boxtimes \tilde{y}) \star \tilde{t}^{>j}\Big) \bullet \Big((t_1^{\leq j} \boxplus y) \cdot t_1^{>j}\Big) \\ &= (t^{\leq j} \boxtimes y) \star t^{>j} \;. \end{aligned}$$

$\boxed{QED}$

Lemma 5. *Let $m \in \mathbb{N}$, $w \in W$ and $t \in S\overset{\propto}{T}$. Then the following conditions are equivalent:*

(i) $w \in S\overset{\propto}{W}^{mm}$,

(ii) $\mathrm{z}(w,t) \in S\overset{\geq}{\mathcal{W}}^{mm}$,

(iii) $\mathrm{s}(w,t) \in S\overset{<}{\mathcal{W}}^{mm}$.

Proof. For $m = 1$, the asserted equivalence follows from Lemma 3. Let $m > 1$ and choose $u, v \in W$ such that $w = u \cdot v$ and $|u| = m = |v|$. First, all three conditions (i), (ii) and (iii) imply that $u, v \in W^{\propto}$, by definition or, resp., Theorem 2. Furthermore, in each case, we have:
$(*)$ v enters the first row $(t \boxtimes u)_1$ of $t \boxtimes u$.
For, in the case that (i) holds, this follows from Proposition 4(b) and $u_i \not\propto v_i$ for all $i \in \underline{m}$, while, assuming (ii), $(*)$ follows from $\mathrm{z}(v_i, t \boxtimes u \cdot v^{<i}) > \mathrm{z}(u_i, t \boxtimes u^{<i})$, that is, $\mathrm{z}(v_i, t \boxtimes u \cdot v^{<i}) \geq 2$ for all $i \in \underline{m}$. Finally, (iii) implies that

$$\mathrm{s}(v_i, t \boxtimes u \cdot v^{<i}) \leq \mathrm{s}(u_i, t \boxtimes u^{<i}) \leq |(t \boxtimes u \cdot v^{<i})_1|$$

for all $i \in \underline{m}$ and thus also $(*)$.

Let $t = t_l \bullet \cdots \bullet t_1$. We consider two cases:

case 1: u enters t_1.

Then w enters t_1, by $(*)$. Applying Lemma 4 and (16), we obtain inductively

$$\text{(i)} \iff t_1 \measuredangle w \in S\overset{\propto}{W}^{mm} \iff \mathrm{z}(t_1 \measuredangle w, t_{>1}) \in S\overset{\geq}{\mathcal{W}}^{mm} \iff \text{(ii)}.$$

and the corresponding equivalence for $\mathrm{s}(w,t)$.

case 2: u passes t_1.

Then Proposition 5(a) particularly implies that $t \boxtimes u = (t \boxtimes u^{<m}) \star u_m$, and there exist $\tilde{w} \in W$ and $x \in \{v_1, \ldots, v_{m-1}, u_m\}$ such that $((t \boxtimes u) \boxtimes v^{<m})_1 = \tilde{w} \cdot x$. As v_m enters $\tilde{w} \cdot x$ whenever (i), (ii) or (iii) holds, by $(*)$, we obtain step by step $x \notin \{v_1, \ldots, v_{m-1}\}$, $x = u_m$ and finally $u_m \not\propto v_m$. Hence, each of the three conditions implies that

$$(t \boxtimes u) \boxtimes v^{<m} = \Big(t \boxtimes (u^{<m} \cdot v^{<m})\Big) \star u_m$$

and

$$\mathrm{z}(v^{<m}, t \boxtimes u) = \mathrm{z}(v^{<m}, t \boxtimes u^{<m}), \qquad \mathrm{s}(v^{<m}, t \boxtimes u) = \mathrm{s}(v^{<m}, t \boxtimes u^{<m}) \ ,$$

by Proposition 5(b). The equivalence of (i), (ii) and (iii) again follows by induction. $\boxed{QED}$

Theorem 3. *Let $t \in S\overset{\propto}{T}$, $q, r \in \mathcal{W}$ and assume that q is a partition. Then, for all $w \in W$, the following three conditions are equivalent:*

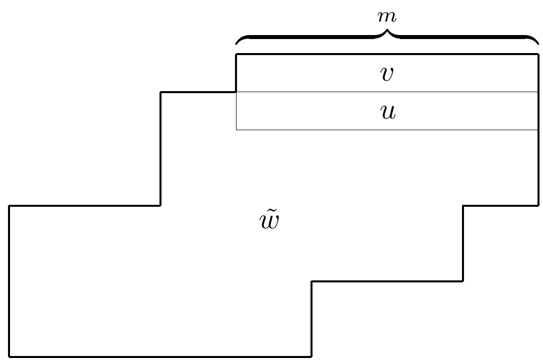
(i) $w \in S^q\overset{\propto}{W}{}^r$,

(ii) $\mathrm{z}(w,t) \in S^q\overset{\geq}{\mathcal{W}}{}^r$,

(iii) $\mathrm{s}(w,t) \in S^q\overset{<}{\mathcal{W}}{}^r$.

Note that, in the special case of $X = \mathbb{N}$, $\propto = \geq$, the preceding theorem implies Theorem 1 in [17], as will be demonstrated after Lemma 6.

Proof of the theorem. Let $n := |w| > 0$, $z = z_1 \cdots z_n := \mathrm{z}(w,t)$, $s = s_1 \cdots s_n := \mathrm{s}(w,t)$ and $r = r_1 \cdots r_l$. For $l = 1$, the asserted equivalence is immediate from Theorem 2. Let $l > 1$. Then, by (7), each of the conditions (i), (ii) and (iii) implies that $q_1 + r_1 \geq q_2 + r_2$. If $q_1 + r_1 > q_2 + r_2$, the equivalence of (i), (ii) and (iii) inductively follows from the equivalence for $w_1 \cdot \ldots \cdot w_{n-1}$ and from Theorem 2. Hence we may assume that $q_1 + r_1 = q_2 + r_2$. Now put $m := r_1$, $\tilde{w} := w_1 \cdot \ldots \cdot w_{n-2m}$, $u := w_{n-2m+1} \cdot \ldots \cdot w_{n-m}$ and $v := w_{n-m+1} \cdot \ldots \cdot w_n$, visualized as follows:

$\mathrm{Tab}_r(w) \quad \sim$ 

Let $\tilde{t} := t \boxplus \tilde{w}$. By Lemma 5, we have

$$u \cdot v \in S\overset{\propto}{W}^{mm} \iff \mathrm{z}(u \cdot v, \tilde{t}) \in S\overset{\geq}{\mathcal{W}}^{mm} \iff \mathrm{s}(u \cdot v, \tilde{t}) \in S\overset{<}{\mathcal{W}}^{mm}.$$

Furthermore, putting $\tilde{r} := r_2 \cdots r_l$ and $\tilde{q} := q_2 \cdots q_{|q|}$, we obtain inductively

$$\tilde{w} \cdot u \in S^{\tilde{q}}\overset{\propto}{W}^{\tilde{r}} \iff \mathrm{z}(\tilde{w} \cdot u, t) \in S^{\tilde{q}}\overset{\geq}{\mathcal{W}}^{\tilde{r}} \iff \mathrm{s}(\tilde{w} \cdot u, t) \in S^{\tilde{q}}\overset{<}{\mathcal{W}}^{\tilde{r}}.$$

Combining these two chains of equivalence, we are done. $\boxed{QED}$

4 Conjugate and Rotated Tableaux

Conjugating or rotating a shifted standard tableau leads again to a shifted standard tableau (with respect to a suitable ordering). This simple observation combined with Theorems 1 and 3 yields our main combinatorial result (Main Theorem 1). For example, in the case of $X = \mathbb{N}$ and $\propto = \leq$, we may consider the partition $q = 11$ and the tableau $t = (2 \cdot 4) \bullet (3 \cdot 3) \bullet (1 \cdot 1 \cdot 1 \cdot 2) \in S^q\overset{\leq}{\mathcal{T}}^{422}$, visualized by

$$t \sim \begin{array}{ccccc} & 1 & 1 & 1 & 2 \\ & 3 & 3 & & \\ 2 & 4 & & & \end{array} .$$

The corresponding q-conjugate and rotated tableau may then be visualized by

$$t^{\star(q)} \sim \begin{array}{ccc} & & 2 \\ 1 & 3 & 4 \\ 1 & 3 & \\ 1 & & \\ 2 & & \end{array} \quad \text{and} \quad t^{\prime} \sim \begin{array}{ccccc} & & & 4 & 2 \\ & & 3 & 3 & \\ 2 & 1 & 1 & 1 & \end{array} .$$

Hence $t^{\star(q)}$ and $t^{\shortmid}$ are indeed shifted standard tableaux with respect to $<$ and $\geq$, resp. More precisely, we have

$$t^{\star(q)} \in S^2\overset{<}{\mathcal{T}}^{13211} \quad \text{and} \quad t^{\shortmid} \in S^{32}\overset{\geq}{\mathcal{T}}^{224} \quad .$$

In order to analyze these phenomena in general, we need some definitions.

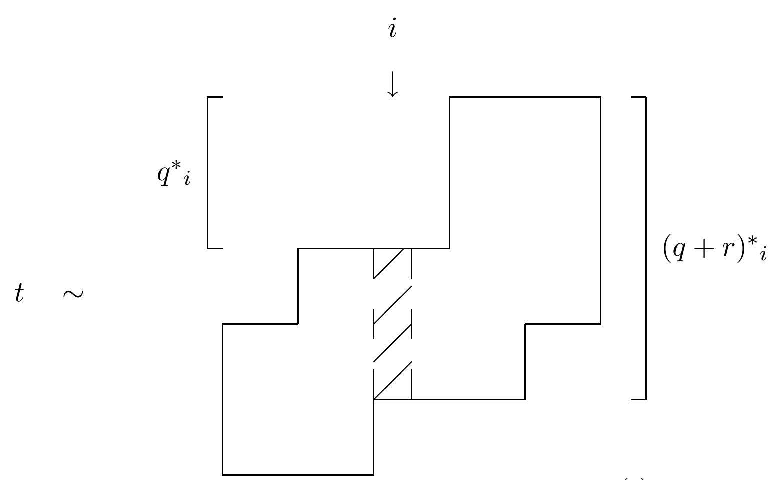
Let $p = p_1 \cdots p_l \in \mathcal{W}$ be a partition. For the *conjugate partition* $p^* = {p^*}_1 \cdots {p^*}_{p_1}$ of p, defined by

$${p^*}_i := |\{\, j \in \underline{l} \,|\, p_j \geq i \,\}|$$

for all $i \in \underline{p_1}$, we then have $(p^*)^* = p$. Now let $q, r \in \mathcal{W}$. We define words $q+r, q-r \in \mathcal{W}$ by $(q+r)_\infty = q_\infty + r_\infty$ and $(q-r)_\infty = q_\infty - r_\infty$, resp., where, in the second case, nonpositive integers in $q-r$ are omitted. Let $r = r_1 \cdots r_l$ and $t = t_l \bullet \cdots \bullet t_1 \in T^r$. Assume that q and $q+r$ are partitions. For all $i \in \underline{q_1 + r_1}$, we put $\nu_1 := \begin{cases} {q^*}_i + 1 & , \quad i \leq q_1 \\ 1 & , \quad i > q_1 \end{cases}$, $\nu_2 := (q+r)^*{}_i$ and $s_i := t_{\nu_1, i-q_{\nu_1}} \cdot t_{\nu_1+1, i-q_{\nu_1+1}} \cdot \ldots \cdot t_{\nu_2, i-q_{\nu_2}}$, where $q_\nu := 0$ whenever $\nu > |q|$. Now we define

$$t^{\star(q)} := s_{q_1+r_1} \bullet \cdots \bullet s_1 \quad .$$

The i-th row s_i of the *q-conjugate tableau* $t^{*(q)}$ of t may be visualized as follows:



If q is the empty partition, we write $t^\star$ instead of $t^{\star(q)}$. As an immediate consequence of the definitions, we obtain:

Proposition 6. *Let $q, r \in \mathcal{W}$ such that $|q| \leq |r|$. Let $t \in T^r$. Assume that q and $q+r$ are partitions. Then we have $t^{\star(q)} \in T^{(q+r)^* - q^*}$ and*

$$\left(t^{\star(q)}\right)^{\star(q^*)} = t \quad .$$

The *rotated tableau* $t^{\shortmid}$ of t is defined by

$$t^{\shortmid} := \overline{t_1} \bullet \cdots \bullet \overline{t_l} \quad ,$$

where we have used the notation $\overline{w} := w_1 \,\bar{\cdot}\, \ldots \,\bar{\cdot}\, w_n$ for all $w = w_1 \cdot \ldots \cdot w_n \in W$. Obviously, we have $(t^{\shortmid})^{\shortmid} = t$ for all $t \in T$. For all $q, u \in \mathcal{W}$, we write

$$q \preceq u \quad ,$$

if $(q_\infty)_i \leq (u_\infty)_i$ for all $i \in \mathbb{N}$. Let q and u be partitions such that $q \preceq u$. Let $m := |u|$. Then we put

$$(u, q)' := (u', q') \quad ,$$

where u', q' are the unique partitions such that

$$(u'_\infty)_i := \begin{cases} u_1 - (q_\infty)_{m+1-i} & , \quad i \leq m \\ 0 & , \quad i > m \end{cases}$$

and

$$(q'_\infty)_i := \begin{cases} u_1 - (u_\infty)_{m+1-i} & , \quad i \leq m \\ 0 & , \quad i > m \end{cases}$$

for all $i \in \mathbb{N}$. It is easy to see now that $q' \preceq u'$,

$$(u^*, q^*)' = (u'^*, q'^*) \tag{20}$$

and, in the case of $|q| < |u|$, $\Big((u, q)'\Big)' = (u, q)$.

Lemma 6. *Let $q, u \in \mathcal{W}$ be partitions such that $q \preceq u$. Let $u', q' \in \mathcal{W}$ such that $(u, q)' = (u', q')$. Then, for any $t \in T$, the following conditions are equivalent:*

(i) $t \in S^q \overset{\propto}{T}{}^{u-q}$,

(ii) $t^{\star(q)} \in S^{q^*} \overset{\cdot\propto}{T}{}^{u^*-q^*}$,

(iii) $t^{\shortmid} \in S^{q'} \overset{\overline{\propto}}{T}{}^{u'-q'}$,

(iv) $t^{\star(q)\shortmid} \in S^{q'^*} \overset{\overline{\cdot\propto}}{T}{}^{u'^*-q'^*}$.

As an immediate consequence of this lemma, we observe that, for the set $\mathcal{F}_{q,u}$ defined in [17], we have

$$w \in \mathcal{F}_{q,u} \iff w \in S^{q'} \overset{\overline{\geq}}{T}{}^{u'-q'}$$

for any $w \in \mathcal{W}$. Hence Theorem 1 in [17] indeed follows from our Theorem 3, if we take into account that, for the *template tableau* $T(w)$ defined in [17], we have $T(w)\mathfrak{z} = Q_{\geq}(w)$.

Proof of the lemma. We may assume that $|q| < m := |u|$. For, otherwise, we may consider $\tilde{q}, \tilde{u} \in \mathcal{W}$, defined by $(\tilde{q}_\infty)_i := (q_\infty)_i - q_m$ and $(\tilde{u}_\infty)_i := (u_\infty)_i - q_m$ for all $i \in \mathbb{N}$, instead of q, u. The corresponding sets of standard tableaux in (i)-(iv) remain the same.
The equivalence of (i) and (ii) is immediate from the definitions. In order to prove the equivalence of (i) and (iii), it suffices to show that (i) implies (iii), as $(t^{!})^{!} = t$ and $((u,q)')' = (u,q)$. Let $t \in S^q\overset{\propto}{T}{}^{u-q}$ and $r = r_1 \cdots r_l := u - q$. Then we have $u' - q' = r_l \cdots r_1$. By definition, it follows that

$$t' = \overline{t_1} \bullet \cdots \bullet \overline{t_l} \in \overset{\overline{\propto}}{T}{}^{\overline{r}} = \overset{\overline{\propto}}{T}{}^{u'-q'}.$$

Hence, for the proof of (iii), it remains to be shown that, for all $i \in \underline{l-1}$,

$$t_i^{!} \cdot \overline{\propto}_{q_i'-q_{i+1}'} t_{i+1}^{!} \quad .$$

As $\cdot\overline{\propto} = \not\propto$, this is equivalent to

$$\overline{t_{i+1}} \not\propto_{q_{l-i}'-q_{l-i+1}'} \overline{t_i}$$

for all $i \in \underline{l-1}$. But, for all $i \in \underline{l-1}$, we have $t_i \overline{\not\propto}_{q_i-q_{i+1}} t_{i+1}$, by (i), and furthermore

$$|t_i| + q_i - q_{i+1} = u_i - q_{i+1} = |t_{i+1}| - u_{i+1} + u_i = |t_{i+1}| + q_{l-i}' - q_{l+1-i}'.$$

Hence (iii) follows from the following easy seen equivalence: If $k, l, m, n \in \mathbb{N}$ such that $n + l = k + m$ and if $v, w \in W$ such that $|v| = n$, $|w| = m$, we have

$$w \overline{\not\propto}_k v \iff \overline{v} \not\propto_l \overline{w} \quad .$$

Finally, the equivalence of (ii) and (iv) follows from the equivalence of (i) and (iii), by (20). $\boxed{QED}$

In the case of $X = \mathbb{N}$, we obtain from Lemma 6:

Corollary 2. *Let $p, q, r \in \mathcal{W}$ such that q and $u := q + p$ are partitions. Let $q', u' \in \mathcal{W}$ such that $(u,q)' = (u', q')$ and put*

$$q_{\leq} := q, \quad q_{\geq} := q', \quad q_{<} := q^* \quad \text{and} \quad q_{>} := q'^*$$

and

$$p_{\leq} := p, \quad p_{\geq} := u' - q', \quad p_{<} := u^* - q^* \quad \text{and} \quad p_{>} := u'^* - q'^*.$$

Then we have

$$|S^q\overset{\leq}{\mathcal{T}}{}^p_r| = |S^{q\propto}\overset{\propto}{\mathcal{T}}{}^{p\propto}_r|$$

for all $\propto \in \{\leq, \geq, <, >\}$. *In particular, we have* $|S\overset{\leq}{\mathcal{T}}{}^p_r| = |S\overset{<}{\mathcal{T}}{}^{p^*}_r|$ *and* $|S\overset{\geq}{\mathcal{T}}{}^p_r| = |S\overset{>}{\mathcal{T}}{}^{p^*}_r|$.

Proof. Conjugating and rotating induce bijections between the sets in question, by Lemma 6. $\boxed{QED}$

Remark 1. Let $n \in \mathbb{N}$, $p \vdash n$ and $r = r_1 \cdots r_l \vdash n$. For any $t \in S\overset{\geq}{\mathcal{T}}{}^p_r$, we obtain a tableau $\tilde{t} \in S\overset{\leq}{\mathcal{T}}{}^p_{\overline{r}}$ by replacing the letters i in t by $l+1-i$, for all $i \in \underline{l}$. The mapping $t \longmapsto \tilde{t}$ induces an involution between the sets $S\overset{\geq}{\mathcal{T}}{}^p_r$ and $S\overset{\leq}{\mathcal{T}}{}^p_{\overline{r}}$. As $\overline{r} = r_l \cdots r_1$ is a rearrangement of r, we have $|S\overset{\leq}{\mathcal{T}}{}^p_{\overline{r}}| = |S\overset{\leq}{\mathcal{T}}{}^p_r|$ (for a simple combinatorial proof, see [15], Theorem 7.10.2). Hence

$$|S\overset{\geq}{\mathcal{T}}{}^p_r| = |S\overset{\leq}{\mathcal{T}}{}^p_r|. \tag{21}$$

For any tableau $t \in SY\mathcal{T}$, the *column word* of t is defined by

$$t\mathfrak{s} := t^*\mathfrak{z} \quad ,$$

that is, the i-th letter of $t\mathfrak{s}$ is the number of the column of t containing i, for all i.

Corollary 3. *Let* $t \in SY\mathcal{T}$ *and* $q, p \in \mathcal{W}$. *Assume that* q *and* $q+p$ *are partitions. Then we have*

$$t\mathfrak{z} \in S^q\overset{\geq}{\mathcal{W}}{}^p \iff t\mathfrak{s} \in S^q\overset{<}{\mathcal{W}}{}^p.$$

Proof. By Theorem 1, we can find a word $w \in \mathcal{W}$ such that $(P_{\leq}(w), Q_{\leq}(w)) = (t, t\mathfrak{z})$, for there exists a partition $r \in \mathcal{W}$ such that $t \in S\overset{\leq}{\mathcal{T}}{}^r$ and $t\mathfrak{z} \in \mathcal{L}_r$. Furthermore, we have $t\mathfrak{z} = \mathrm{z}(w, \ddot{\imath})$, by (17), and hence $t\mathfrak{s} = \mathrm{s}(w, \ddot{\imath})$ by induction. Applying Theorem 3, we obtain

$$z\mathfrak{z} = \mathrm{z}(w, \ddot{\imath}) \in S^q\overset{\geq}{\mathcal{W}}{}^p \iff w \in S^q\overset{\leq}{\mathcal{W}}{}^p \iff t\mathfrak{s} = \mathrm{s}(w, \ddot{\imath}) \in S^q\overset{<}{\mathcal{W}}{}^p.$$

$\boxed{QED}$

We are now in a position to prove our main combinatorial result.

Main Theorem 1. *Let* $p, q, r \in \mathcal{W}$ *such that* q *and* $q+p$ *are partitions. Then we have*

$$|S^q\overset{\propto}{T}{}^p_r| = |\bigcup_{u \text{ partition}} S\overset{\propto}{T}{}^u_r \times S^q\overset{\geq}{\mathcal{L}}{}^p_u| = |\bigcup_{u \text{ partition}} S\overset{\propto}{T}{}^u_r \times S^q\overset{<}{\mathcal{L}}{}^p_{u^*}|$$

PROOF. The first identity is a combination of Theorems 1 and 3. Let $u \in \mathcal{W}$ be a partition. Then the mapping $t \longmapsto t\mathfrak{s}$ induces a bijection of $SY\mathcal{T}^u$ onto $\mathcal{L}_{u^*}$, by Corollary 2 and (14). Hence, the mapping $\varphi : \mathcal{L}_u \longrightarrow \mathcal{L}_{u^*}$, $w \longmapsto w\mathfrak{z}^{-1}\mathfrak{s}$ is bijective, and $S^q \overset{\geq}{\mathcal{L}}{}^p_u \varphi = S^q \overset{<}{\mathcal{L}}{}^p_{u^*}$ for all q, p, by Corollary 3. This implies the second identity. $\boxed{QED}$

5 An Eightfold Littlewood-Richardson Theorem

Let K be a field of characteristic 0. For all $n \in \mathbb{N}$, we denote by S_n the symmetric group on $\underline{n}$ and by $\mathrm{Cl}_K(S_n)$ the ring of class functions of S_n over K. For all χ, $\psi \in \mathrm{Cl}_K(S_n)$, we write

$$(\chi, \psi)_{S_n} := \frac{1}{n!} \sum_{\sigma \in S_n} \chi(\sigma)\psi(\sigma^{-1})$$

for the ordinary scalar product of χ and ψ. Let $q = q_1 \cdots q_k \in \mathcal{W}$ such that $q_1 + \cdots + q_k = n$. Inducing the trivial character of a Young subgroup of S_n of type q, we obtain a character ξ^q of S_n which is called *Young character* corresponding to q. We then have $\xi^q = \xi^r$ whenever q is a rearrangement of r. For any $p \vdash n$, we denote by ζ^p the irreducible character of S_n corresponding to p. Then, in particular, $\mathrm{sgn}_n := \zeta^{11\ldots1} \in \mathrm{Cl}_K(S_n)$ is the *sign character* of S_n, and we have

$$\mathrm{sgn}_n\, \zeta^p = \zeta^{p^*} \tag{22}$$

(see [6], 4.3.14). The *Kostka matrix* $\mathcal{K}_n = (k_{qp})_{q,p \vdash n}$ is defined by

$$\xi^q = \sum_{p \vdash n} k_{qp}\, \zeta^p \tag{23}$$

for all $q \vdash n$, that is, k_{qp} is the multiplicity $(\xi^q, \zeta^p)_{S_n}$ of the irreducible character corresponding to p in the Young character corresponding to q. Row and column indices of $\mathcal{K}_n$ are assumed to be arranged in lexicographic decreasing order. Finally, for all $p, q, r \in \mathcal{W}$, we put

$$st^p_r := |S\overset{\leq}{\mathcal{T}}{}^p_r| \quad \text{and} \quad s^q t^p_r := |S^q \overset{\leq}{\mathcal{T}}{}^p_r|.$$

Then we have (see [6], 4.3.22, 4.4.6):

Theorem 4. *Let $n \in \mathbb{N}$.*

(a) *The Kostka matrix $\mathcal{K}_n$ is lower triangular with units in the diagonal. Particularly, $\{\, \xi^q \mid q \vdash n \,\}$ is a K-basis of* $\mathrm{Cl}_K(S_n)$.

(b) *For all $q, p \vdash n$, we have $k_{qp} = st_q^p$.*

Let

$$\mathcal{C} := \bigoplus_{n \in \mathbb{N}_0} \mathrm{Cl}_K(S_n).$$

Then, by Theorem 4(a), $\{\, \xi^q \mid q \in \mathcal{W},\ q \text{ partition} \,\}$ is a K-basis of $\mathcal{C}$. Hence, the *outer product* $\bullet$ on $\mathcal{C}$ may be defined by

$$\xi^q \bullet \xi^r := \xi^{qr}$$

for all partitions q, r, and bilinearity. We are aiming at combinatorial descriptions of the *Littlewood-Richardson (L-R) coefficients*

$$c_{qp}^u := (\zeta^q \bullet \zeta^p, \zeta^u)_{S_{n+k}}$$

for all $n, k \in \mathbb{N}$, $p \vdash k$, $q \vdash n$ and $u \vdash n+k$. Our starting point is the following well-known result due to Young [18]. For the reader's convenience, a short proof is given.

Theorem 5 (Young's Rule). *For all $k, n \in \mathbb{N}$, $p \vdash n$, $r \vdash k$, we have*

$$\zeta^p \bullet \xi^r = \sum_{u \vdash n+k} s^p t_r^{u-p}\, \zeta^u.$$

Proof. Let $q = q_1 \cdots q_l \vdash n$ and $u \vdash n+k$. For all $w = w_1 \cdots w_k \in \mathcal{W}$, we define $w^{+l} := (w_1 + l) \cdots (w_k + l)$. For all $t = t_m \bullet \cdots \bullet t_1 \in \mathcal{T}$, we define $t^{+l} := t_m^{+l} \bullet \cdots \bullet t_1^{+l}$. Then the mapping

$$\bigcup_{p \vdash n} S\overset{\leq}{\bar{\mathcal{T}}}{}_q^p \times S^p\overset{\leq}{\bar{\mathcal{T}}}{}_r^{u-p} \longrightarrow S\overset{\leq}{\bar{\mathcal{T}}}{}_{qr}^u,\ (s,t) \longmapsto s \star t^{+l}$$

is bijective. Hence we have $\sum_{p \vdash n} st_q^p\ \ s^p t_r^{u-p} = st_{qr}^u$. By Theorem 4(b), this implies the identity

$$\mathcal{K}_n (s^p t_r^{u-p})_{p \vdash n,\, u \vdash n+k} = (st_{qr}^u)_{q \vdash n,\, u \vdash n+k}$$

or, equivalently, $(s^p t_r^{u-p})_{p \vdash n,\, u \vdash n+k} = \mathcal{K}_n^{-1} (st_{qr}^u)_{q \vdash n,\, u \vdash n+k}$. Writing $\mathcal{K}_n^{-1} = (e_{pq})$ and applying Theorem 4(b) again we may conclude that

$$\zeta^p \bullet \xi^r \;=\; \sum_{q \vdash n} e_{pq}\, \xi^{qr} = \sum_{u \vdash n+k} \big(\sum_{q \vdash n} e_{pq} st_{qr}^u\big)\, \zeta^u = \sum_{u \vdash n+k} s^p t_k^{u-p}\, \zeta^u.$$

$\boxed{QED}$

Corollary 4. *Let $n, k \in \mathbb{N}$. For all $p \vdash k$, $q \vdash n$ and $u \vdash n+k$, let C^u_{qp} be a set such that there exists a bijection*

$$S^q\overset{\leq}{T}{}^{u-q}_p \longrightarrow \bigcup_{r\vdash k} S\overset{\leq}{T}{}^r_p \times C^u_{qr} \quad .$$

Then, for all $p \vdash k$, $q \vdash n$ and $u \vdash n+k$, we have $c^u_{qp} = |C^u_{qp}|$. In particular, we have $q, p \preceq u$ whenever $c^u_{qp} \neq 0$.

Proof. Let $q \vdash n$ and $p \vdash k$. By Theorem 5 and Theorem 4(b), we have

$$\sum_{u\vdash n+k} s^q t^{u-q}_p \, \zeta^u = \zeta^q \bullet \xi^r = \sum_{r\vdash k} st^r_p \, \zeta^q \bullet \zeta^r = \sum_{u\vdash n+k} \sum_{r\vdash k} st^r_p \, c^u_{qr} \, \zeta^u.$$

Comparing the coefficients of ζ^u on both sides, we obtain

$$\sum_{r\vdash k} st^r_p \, |C^u_{qr}| = s^q t^{u-q}_p = \sum_{r\vdash k} st^r_p \, c^u_{qr}$$

for all $u \vdash n+k$. Hence, $c^u_{qp} \neq 0$ implies $s^q t^{u-q}_p \neq 0$ and therefore $q \preceq u$. Now $p \preceq u$ follows from $\zeta^p \bullet \zeta^q = \zeta^q \bullet \zeta^p$. As p and q have been chosen arbitrarily, we obtain furthermore

$$(c^u_{qp})_{\substack{u\vdash n+k\\ p\vdash k}} (st^r_p)_{\substack{r\vdash k\\ p\vdash k}} = (|C^u_{qp}|)_{\substack{u\vdash n+k\\ p\vdash k}} (st^r_p)_{\substack{r\vdash k\\ p\vdash k}} \qquad \text{for all } q \vdash n.$$

This identity of matrices implies our claim, by the regularity of the (transposed) Kostka matrix and Theorem 4(b). $\boxed{QED}$

Now we are in a position to give eight combinatorial descriptions of the L-R coefficients.

Main Theorem 2. *Let $n, k \in \mathbb{N}$, $q \vdash n$, $p \vdash k$, $u \vdash n+k$ and $u', q' \in \mathcal{W}$ such that $(u, q)' = (u', q')$. Then the Littlewood-Richardson coefficient c^u_{qp} is equal to the number of q- (q'-, q^*-, q'^*-, resp.) shifted standard tableaux t with respect to $\leq$ of shape $u - q$ ($u' - q'$, $u^* - q^*$, $u'^* - q'^*$, resp.) such that the word w obtained by reading off the entries of t row-wise from top right to bottom left is a standard word (= "lattice permutation") of content p (p, p^*, p^*, resp.).*

(In this description, the word "row-wise" may be replaced by "column-wise".)

The first variation listed above is the classical description due to Littlewood and Richardson [9].

Proof of the theorem. By Corollary 4, we may assume that $q \preceq u$. For all partitions r, we put

$$r(\leq) = r(\geq) := r \quad \text{ and } \quad r(<) = r(>) := r^* \quad . \tag{24}$$

Then we have $|S\overset{\propto}{\mathcal{T}}{}^r_p| = |S\overset{\leq}{\mathcal{T}}{}^{r(\propto)}_p|$ for all $\propto \in \{\leq, \geq, <, >\}$, by (21) and Corollary 2. Let $v := u - q$. Applying Corollary 2 again and defining $q_\propto$, $v_\propto$ accordingly, for all $\propto \in \{\leq, \geq, <, >\}$, we obtain

$$|S^q\overset{\leq}{\mathcal{T}}{}^v_p| \;=\; |S^{q_\propto}\overset{\propto}{\mathcal{T}}{}^{v_\propto}_p| = |\bigcup_{r\vdash k} S\overset{\propto}{\mathcal{T}}{}^r_p \times S^{q_\propto}\overset{\geq}{\mathcal{L}}{}^{\,v_\propto}_{\,r}| = |\bigcup_{r\vdash k} S\overset{\leq}{\mathcal{T}}{}^r_p \times S^{q_\propto}\overset{\geq}{\mathcal{L}}{}^{\,v_\propto}_{\,r(\propto)}|$$

by the first equality in Main Theorem 1, and

$$|S^q\overset{\leq}{\mathcal{T}}{}^v_p| \;=\; |S^{q_\propto}\overset{\propto}{\mathcal{T}}{}^{v_\propto}_p| = |\bigcup_{r\vdash k} S\overset{\propto}{\mathcal{T}}{}^r_p \times S^{q_\propto}\overset{<}{\mathcal{L}}{}^{\,v_\propto}_{\,r^*}| = |\bigcup_{r\vdash k} S\overset{\leq}{\mathcal{T}}{}^r_p \times S^{q_\propto}\overset{<}{\mathcal{L}}{}^{\,v_\propto}_{\,r(\propto)^*}|$$

by the second equality in Main Theorem 1. Now Corollary 4 implies that

$$(*) \quad c^u_{qp} = |S^{q_\propto}\overset{\geq}{\mathcal{L}}{}^{\,v_\propto}_{\,p(\propto)}| = |S^{q_\propto}\overset{<}{\mathcal{L}}{}^{\,v_\propto}_{\,p(\propto)^*}| \quad \text{for all } \propto \in \{\leq, \geq, <, >\}.$$

In the case of $\propto = \geq$, the first equality in $(*)$ says that c^u_{qp} is the number of standard q'-shifted $\geq$-fillings of shape $u' - q'$ and content p. But this is simply the number of q-shifted standard tableaux t of shape $u - q$ and content p such that the word w obtained by reading off the entries of t row-wise from top right to bottom left is a standard word, that is, the classical description of c^u_{qp}. The remaining seven descriptions may be obtained from $(*)$ analogously. $\boxed{QED}$

Let $n, k \in \mathbb{N}$ and $q \vdash n$, $u \vdash n + k$ such that $q \preceq u$. The *skew character* $\zeta^{u/q} \in \mathrm{Cl}_K(S_k)$ of S_n corresponding to u and q may be defined by

$$\zeta^{u/q} := \sum_{p\vdash k} c^u_{qp}\, \zeta^p \quad ,$$

or, equivalently, by

$$(\zeta^{u/q}, \zeta^p)_{S_k} = (\zeta^u, \zeta^q \bullet \zeta^p)_{S_k} \quad \text{for all } p \vdash k.$$

As a consequence of Main Theorem 2, we obtain the following identities of skew characters:

Corollary 5. *Let $n, k \in \mathbb{N}$, $q \vdash n$, $u \vdash n + k$ such $q \preceq u$. Then we have*

$$\zeta^{u/q} = \mathrm{sgn}_k\, \zeta^{u^*/q^*} = \zeta^{u'/q'} = \mathrm{sgn}_k\, \zeta^{u'^*/q'^*} \quad ,$$

where $q', u' \in \mathcal{W}$ such that $(u, q)' = (u', q')$.

The first equality in Corollary 5 is due to Aitken [1], while the identity $\zeta^{u/q} = \zeta^{u'/q'}$ is a special case of a symmetry property of L-R coefficients due to Berenstein and Zelevinsky [2].

Proof of the corollary. Let $p \vdash k$. Then, for all $\chi \in \mathrm{Cl}_K(S_k)$, we have

$$(\mathrm{sgn}_k\, \chi, \zeta^p)_{S_k} = (\chi, \mathrm{sgn}_k\, \zeta^p)_{S_k} = (\chi, \zeta^{p^*})_{S_k} \quad ,$$

by (22). Hence our assertions are equivalent to $c^u_{qp} = c^{u^*}_{q^*p^*} = c^{u'}_{q'p} = c^{u'^*}_{q'^*p^*}$. These identities are immediate from Main Theorem 2. $\boxed{QED}$

6 Bijections between Littlewood-Richardson sets

In Main Theorem 2, for all $q \vdash n$, $r \vdash k$, $u \vdash n+k$, eight different sets were introduced with cardinality c^u_{qr}, which will be referred to as Littlewood-Richardson (L-R) sets in the sequel. In the proof of Main Theorem 2, a combinatorial connection was established between each pair of families of L-R sets: For, if C^u_{qr} and $\tilde{C}^u_{qr}$ ($q \vdash n$, $r \vdash k$, $u \vdash n+k$) are two such families, we obtain bijections

$$\begin{array}{ccccc} \bigcup_{r\vdash k} S\overset{\leq}{\mathcal{T}}{}^r_p \times C^u_{qr} & \longrightarrow & S^q\overset{\leq}{\mathcal{T}}{}^{u-q}_p & \longrightarrow & \bigcup_{r\vdash k} S\overset{\leq}{\mathcal{T}}{}^r_p \times \tilde{C}^u_{qr} \\ (s,w) & \longmapsto & t & \longmapsto & (\tilde{s},\tilde{w}) \end{array} \tag{25}$$

for all q, u, by Corollary 2 and Main Theorem 1. From a combinatorial point of view, it is natural to ask for a direct bijection $C^u_{qr} \longrightarrow \tilde{C}^u_{qr}$ for all q, u, r. Surprisingly, such a bijection may be obtained as a suitable restriction of the bijection (25), namely by fixing the first component $s \in S\overset{\leq}{\mathcal{T}}{}^r_p$ in (25). It will be shown indeed that for every $s \in S\overset{\leq}{\mathcal{T}}{}^r_p$ there exists $\tilde{s} \in S\overset{\leq}{\mathcal{T}}{}^r_p$ such that the set $\{s\} \times C^u_{qr}$ is mapped onto $\{\tilde{s}\} \times \tilde{C}^u_{qr}$ by the bijection (25).

For the remainder of this section, we fix $n,k \in \mathbb{N}$, $q \vdash n$, $r = r_1 \cdots r_l \vdash k$ and $u \vdash n+k$ such that $q \preceq u$. Furthermore, we put $p := u - q$ and choose $s \in SY\mathcal{T}^r$ arbitrarily. Let $\propto \in \{\leq, <, \geq, >\}$. Bearing in mind the definitions of $q_\propto$ and $p_\propto$ in Corollary 2 and of $r(\propto)$ in (24), we put

$$C_\propto := S^{q_\propto}\overset{\geq}{\mathcal{L}}{}^{\,p_\propto}_{\,r(\propto)} \quad \text{and} \quad \tilde{C}_\propto := S^{q_\propto}\overset{<}{\mathcal{L}}{}^{\,p_\propto}_{\,r(\propto)^*} \quad .$$

In the sequel, bijections will be established from the set $C_\geq$ (the classical L-R description) onto $C_\propto$ and onto $\tilde{C}_\propto$ ($\propto \in \{\leq, <, \geq, >\}$) .

For all $t = t_l \bullet \cdots \bullet t_1 \in \mathcal{T}$, we define

$$\mathrm{word}(t) := t_l \cdot \cdots \cdot t_1 \quad .$$

Then, for all $w \in C_\geq$, there exists a unique (Young) tableau $t_w \in S^{q'}\overset{\leq}{\mathcal{T}}{}^{u'-q'}$ such that

$$\Big(P_\leq(\mathrm{word}(t_w)), Q_\leq(\mathrm{word}(t_w))\Big) = (s,w) \quad ,$$

by Main Theorem 1. Furthermore, defining

$$t^\leq := t, \quad t^\geq := t^{\shortmid}, \quad t^< := t^{\star(q')} \quad \text{and} \quad t^> := t^{\star(q')\shortmid}$$

for any tableau $t \in \mathcal{T}$ and applying Lemma 6, we obtain a bijection

$$S^{q'}\overset{\leq}{\mathcal{T}}{}^{u'-q'} \longrightarrow S^{q'_\propto}\overset{\propto}{\mathcal{T}}{}^{(u'-q')_\propto},\ t \longmapsto t^\propto \quad , \tag{26}$$

for all $\propto \in \{\leq, <, \geq, >\}$.

Theorem 6. *For all* $\propto \in \{\leq, <, \geq, >\}$, *the mappings*

$$C_{\geq} \longrightarrow C_{\propto},\ w \longmapsto Q_{\propto}(word(t_w{}^{\propto}))$$

and

$$C_{\geq} \longrightarrow \tilde{C}_{\propto},\ w \longmapsto \mathrm{s}(word(t_w{}^{\propto}), \ddot{\imath})$$

are bijections.

For the proof we need the following lemma that is based on some well-known properties of the classical Robinson-Schensted correspondence.

Lemma 7. *Let* $t, \tilde{t} \in S^{q'} \overset{\leq}{\mathcal{T}}{}^{u'-q'} \cap Y\mathcal{T}$ *such that* $P_{\leq}(word(t)) = P_{\leq}(word(\tilde{t}))$. *Then, for all* $\propto \in \{\leq, <, \geq, >\}$, *we have*

$$P_{\propto}(word(t^{\propto})) = P_{\propto}(word(\tilde{t}^{\propto})).$$

Proof. Let $Y\mathcal{W}^n$ be the set of all $w = w_1 \cdots w_n \in \mathcal{W}$ such that

$$\{w_1, \ldots, w_n\} = \underline{n}.$$

Then the symmetric group S_n acts on $Y\mathcal{W}^n$ from the left by

$$\pi(w_1 \cdots w_n) := w_{1\pi} \cdots w_{n\pi}$$

and from the right by

$$(w_1 \cdots w_n)\pi := (w_1\pi) \cdots (w_n\pi)$$

for all $\pi \in S_n$, $w = w_1 \cdots w_n \in Y\mathcal{W}^n$. Furthermore, as $\mathrm{word}(t) \in Y\mathcal{W}^n$ for all $t \in Y\mathcal{T}$ such that $|\mathrm{word}(t)| = n$, we obtain a canonical right action of S_n on the set of Young tableaux defined by

$$t\pi := \mathrm{Tab}_r(\mathrm{word}(t)\pi)$$

for all such t and $\pi \in S_n$, where $r \in \mathcal{W}$ such that $r_{\infty} = \mathrm{sh}(t)$. Now let $\varrho \in S_n$ be the order-reversing permutation, defined by $i\varrho = n + 1 - i$ for all $i \in \underline{n}$, and let $w = w_1 \cdots w_n \in Y\mathcal{W}^n$. Then it may be seen easily that

(a) $\mathrm{word}(t^{\mathsf{I}}) = \varrho\, \mathrm{word}(t)$,

(b) $P_{>}(w) = P_{<}(w\, \varrho)\varrho$,

Furthermore, we have:

(c) $P_{<}(\varrho w) = P_{<}(\pi)^{*}$ ([13], Lemma 7),

(d) $P_{<}(w\, \varrho) = P_{<}(w)_{evac}$ ([14], Section 4, 5),

(e) $P_<(\mathrm{word}(t^{*(q')\mathsf{I}})) = P_<(\mathrm{word}(t))$ ([7], Thm. 5, [4], §2, (10)),

where, in (d), $P_<(w)_{evac}$ is Schützenberger's evacutation of the tableau $P_<(w)$. (for details concerning this, and (c), see [8], §4.1.) Now our claim is immediate from (a)-(e): Indeed, for $\propto=\geq$, we obtain

$$\begin{aligned}
P_{\geq}(\mathrm{word}(t^{\geq})) &= P_{\geq}(\mathrm{word}(t^{\mathsf{I}})) = P_{>}(\mathrm{word}(t^{\mathsf{I}})) \\
&= P_{>}(\varrho\mathrm{word}(t)) \quad , \qquad \text{by (a)} \\
&= P_{<}(\varrho\mathrm{word}(t)\,\varrho)\varrho \quad , \qquad \text{by (b)} \\
&= P_{<}(\mathrm{word}(t)\,\varrho)^{*}\varrho \quad , \qquad \text{by (c)} \\
&= (P_{<}(\mathrm{word}(t))_{evac})^{*}\varrho \quad , \qquad \text{by (d)}
\end{aligned}$$

and hence

$$P_{\geq}(\mathrm{word}(t^{\geq})) = (P_{<}(\mathrm{word}(t))_{evac})^{*}\varrho = (P_{<}(\mathrm{word}(\tilde{t}))_{evac})^{*}\varrho = P_{\geq}(\mathrm{word}(\tilde{t}^{\geq})).$$

In the same vein, our claim for $\propto=>$ ($\propto=<$, resp.) follows from

$$\begin{aligned}
P_{>}(\mathrm{word}(t^{>})) &= P_{>}(\mathrm{word}(t^{\star(q')\mathsf{I}})) \\
&= P_{<}(\mathrm{word}(t^{\star(q')\mathsf{I}})\,\varrho)\varrho \quad , \qquad \text{by (b)} \\
&= (P_{<}(\mathrm{word}(t^{\star(q')\mathsf{I}}))_{evac})\,\varrho \quad , \qquad \text{by (d)} \\
&= (P_{<}(\mathrm{word}(t))_{evac})\,\varrho \quad , \qquad \text{by (e)}
\end{aligned}$$

and, resp.,

$$\begin{aligned}
P_{<}(\mathrm{word}(t^{<})) &= P_{<}(\mathrm{word}(t^{\star(q')})) \\
&= P_{<}(\varrho\mathrm{word}(t^{\star(q')\mathsf{I}})) \quad , \qquad \text{by (b)} \\
&= P_{<}(\mathrm{word}(t^{\star(q')\mathsf{I}}))^{*} \quad , \qquad \text{by (c)} \\
&= P_{<}(\mathrm{word}(t))^{*} \quad , \qquad \text{by (e).}
\end{aligned}$$

$\boxed{QED}$

Proof of Theorem 6. Let $\propto\in\{\leq,<,\geq,>\}$ and

$$M := \{\, t \in S^{q'}\overset{\leq}{T}{}^{u'-q'} \mid P_{\leq}(t) = s \,\} \quad .$$

Applying Lemma 7, we can find a tableau $\tilde{s}$ such that $P_{\propto}(\mathrm{word}(t^{\propto})) = \tilde{s}$ for all $t \in M$. Let

$$M_{\propto} := \{\, t \in S^{q'_{\propto}}\overset{\propto}{T}{}^{(u'-q')_{\propto}} \mid P_{\propto}(t) = \tilde{s} \,\} \quad .$$

Then the mapping $M \longrightarrow M_\propto$, $t \longmapsto t^\propto$ is a bijection, by (26) and Lemma 7 again. Furthermore, by Main Theorem 1, the mappings $C_\geq \longrightarrow M$, $w \longmapsto t_w$ and $M_\propto \longrightarrow C_\propto$ ($\tilde{C}_\propto$, resp.), $t \longmapsto Q_\propto(\mathrm{word}(t))$ ($t \longmapsto \mathrm{s}(\mathrm{word}(t), \ddot{\imath})$, resp.) are one to one. This completes the proof. $\boxed{QED}$

The bijections in Theorem 6 should be illustrated by an example: Let $\propto = <$. Then the bijection given in Theorem 6 may be understood as a bijective proof of the identity $c^u_{qr} = c^{u^*}_{q^*r^*}$. Let $n = 3$, $k = 9$, $q = 21$, $r = 432$ and $u = 543$. Then, for the word $w := 111221332$, we have

$$\mathrm{Tab}_{333}(w) \sim \begin{array}{ccccc} & & 3 & 3 & 2 \\ & 2 & 2 & 1 & \\ 1 & 1 & 1 & & \end{array}$$

and hence $w \in S^{q'}\overset{\geq}{\mathcal{L}}{}^{u'-q'}_{r} = C_\geq$. Furthermore, $s = 89 \bullet 567 \bullet 1234 \in SY\mathcal{T}^r$. We obtain

$$t_w \sim \begin{array}{ccccc} & & 2 & 3 & 4 \\ & 5 & 6 & 7 & \\ 1 & 8 & 9 & & \end{array} \quad \text{and} \quad {t_w}^< \sim \begin{array}{ccc} & & 1 \\ & 5 & 8 \\ 2 & 6 & 9 \\ 3 & 7 & \\ 4 & & \end{array} \quad ,$$

hence $\mathrm{word}({t_w}^<) = 437269581$. Now, indeed, we have $\tilde{w} = Q_<(437269581) = 121321324 \in S^{q'^*}\overset{\geq}{\mathcal{L}}{}^{u'^*-q'^*}_{r^*}$, as

$$\mathrm{Tab}_{12321}(121321324) \sim \begin{array}{ccc} & & 1 \\ & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & \\ 4 & & \end{array}$$

is a q^*-shifted standard tableau.

Remark 2. In [5], instead of $C_\geq$ and $C_>$, the sets $C^{u/q}_r := C_\geq \mathfrak{z}^{-1}$ and $C^{u^*/q^*}_{r^*} := C_> \mathfrak{z}^{-1}$ are considered. Using $\alpha_{u/q}$ defined in [5], we find that

$$\mathrm{word}(t^{\star(q')}) = \alpha^{-1}_{u/q}\mathrm{word}(t)$$

for all $t \in S^{q'}\overset{\leq}{T}{}^{u'-q'} \cap Y\mathcal{T}$. Let $w \in C_\geq$ and $T := w\mathfrak{z}^{-1}$. Then, applying

Schützenberger's theorem ([14], section 4) and Theorem 3.11 in [5], we obtain[1]

$$\begin{aligned} Q_<(\mathrm{word}(t_w^{\star(q')}))\mathfrak{z}^{-1} &= P_<(\mathrm{word}(t_w)^{-1}\alpha_{u/q}) \\ &= P_<(\mathrm{word}(t_w)^{-1}\varrho)\alpha_{u/q}\varrho \\ &= (T^\star)_{evac}\alpha_{u/q}\varrho \,. \end{aligned}$$

Thus Theorem 3.14 in [5] is the special case where $\propto = <$ of our Theorem 6. Comparing Example 3.15 in [5] with the example given above might illustrate this.

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